

Constraints on a class of classical solutions in open string field theory

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Abstract

We calculate boundary states for general string fields in the KBc subalgebra under some regularity conditions based on the construction by Kiermaier, Okawa, and Zwiebach. The resulting boundary states are always proportional to that for the perturbative vacuum $|B\rangle$. In this framework, the equation of motion implies that boundary states are independent of the auxiliary parameter s associated with the length of the boundary. By requiring the s -independence, we show that the boundary states for classical solutions in our class are restricted to $\pm|B\rangle$ and 0. In particular, there exist no string fields which reproduce boundary states for multiple D-brane backgrounds. While we know that the boundary states $|B\rangle$ and 0 are reproduced by solutions for the perturbative vacuum and the tachyon vacuum, respectively, no solutions reproducing $-|B\rangle$ have been constructed. In this paper we also propose a candidate for such a solution, which may describe the ghost D-brane.

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1 Introduction and summary

String field theory is a field theoretical approach to non-perturbative aspects of string theory. Its classical solutions describe consistent backgrounds of the string, and it can potentially give a framework to explore various string vacua.

In the case of open string field theory [1], considerable understanding of the landscape has been obtained especially since Schnabl constructed an analytic solution for the tachyon vacuum [2], where unstable D-branes have disappeared and there are no physical excitations of the open string. It is notable that classical open string field theory can describe the decay process of D-branes [3–15] as well as the tachyon vacuum [2, 16, 17] without explicit closed string degrees of freedom. Since closed strings are considered to be emitted in the D-brane decay process, classical open string theory is expected to have some information about the

closed string. By further investigating classical solutions in open string field theory, we would like to explore the scope of open string theory.

Schnabl's original solution was constructed from a class of wedge-based states.¹ When we write the wedge state W_α as $W_\alpha = e^{\alpha K}$,² the solution can be written in terms of the states K , B , and c . These states are associated with the energy-momentum tensor, the b ghost, and the c ghost, respectively.³ They satisfy the following simple algebraic relations called the KBc subalgebra [18]:

$$B^2 = c^2 = 0, \quad \{B, c\} = 1, \quad QB = K, \quad Qc = cKc, \quad [K, B] = 0, \quad (1.1)$$

where Q is the BRST operator of the open bosonic string. These algebraic relations and their extension have been the starting point to construct analytic solutions in open string field theory [3–7, 14, 15, 17, 21–39]. An important property of the KBc subalgebra is that we can define the states without specifying the D-brane configuration at the perturbative vacuum. Furthermore, the algebraic relations (1.1) follow only from the operator product expansions of the energy momentum tensor, the b ghost, and the c ghost. In this sense, the KBc subalgebra is in the universal sector of open bosonic string field theory [40], which is expected to contain classical solutions such as those for the tachyon vacuum and multiple D-branes.

In [18], Okawa proposed a class of formal solutions in the KBc subalgebra as a generalization of Schnabl's solution:

$$\Psi = F(K)c \frac{KB}{1 - F(K)^2} cF(K), \quad (1.2)$$

where $F(K)$ is a function of K . This form of string fields can be formally written in the pure-gauge form,

$$\Psi = U^{-1}QU \quad \text{with} \quad U = 1 - F(K)cBF(K), \quad (1.3)$$

and therefore, they are expected to satisfy the equation of motion of open bosonic string field theory:

$$Q\Psi + \Psi^2 = 0. \quad (1.4)$$

Recently in [36, 39], Murata and Schnabl systematically studied this class of solutions and made an interesting proposal for multiple D-brane solutions. They evaluated energy and a kind of gauge-invariant observables [41–43] and found that those for n D-branes are reproduced by choosing the function $F(K)$ such that

$$\lim_{z \rightarrow 0} z \frac{(F(z)^2)'}{1 - F(z)^2} = 1 - n. \quad (1.5)$$

¹ We denote wedge states [19, 20] with operator insertions by wedge-based states.

² Products of string fields in this paper are defined using Witten's star product [1].

³ A precise definition of these states are given in section 2.

In fact, the proposed solutions are singular and require some regularization. In order to make the solutions fully acceptable, one has to regularize them so that they reproduce physical quantities such as energy and the gauge-invariant observables without ambiguity as well as they respect the equation of motion. However, despite some efforts [39, 44, 45], no regularization method consistent with all the above requirements is known.

In [46], one of the authors evaluated the boundary states for string fields in the form of (1.2) based on the construction by Kiermaier, Okawa, and Zwiebach [47].⁴ It was found that the proposed multiple D-brane solutions do not reproduce the expected boundary states. It was also found that the boundary states non-trivially depend on the parameter s associated with the length of the boundary. This is a serious problem because the non-trivial s -dependence of the boundary states indicates violation of the equation of motion [47]. These results suggest a difficulty in the construction of multiple D-brane solutions in the form of (1.2) without additional terms.

The purpose of this paper is to further develop the argument in [46] and clarify which class of classical solutions can reproduce physically desired boundary states. By requiring the s -independence of the boundary state as a necessary condition to satisfy the equation of motion, we investigate possible solutions in the KBc subalgebra. Extending the calculation in [46], we evaluate the boundary states for general string fields in the KBc subalgebra given by

$$\Psi = \sum_i F_i(K) c B G_i(K) c H_i(K) \quad (1.6)$$

under some regularity conditions introduced in section 2.2. The obtained boundary state⁵ for the string field (1.6) is given by

$$|B_*(\Psi)\rangle = \frac{e^{(x+1)s} - e^{ys}}{e^s - 1} |B\rangle, \quad (1.7)$$

where $|B\rangle$ is the boundary state for the perturbative vacuum and the parameter s is associated with the length of the boundary. The c -numbers x and y are given by

$$x = \sum_i G_i(0) \left(\frac{1}{2} F_i(0) H_i(0) + F'_i(0) H_i(0) \right), \quad (1.8)$$

$$y = \sum_i G_i(0) \left(\frac{1}{2} F_i(0) H_i(0) - F'_i(0) H'_i(0) \right). \quad (1.9)$$

⁴ Recently in [48], another interesting approach to construct boundary states from open string classical solutions was proposed.

⁵ To be precise, the closed string state $|B_*(\Psi)\rangle$ constructed by Kiermaier, Okawa, and Zwiebach corresponds to the boundary state up to BRST-exact terms. Moreover, it does not satisfy some properties of boundary states unless Ψ satisfies the equation of motion. However, in this section, we call it the boundary state for simplicity.

Note that we do not use the equation of motion to derive the formulae (1.7)-(1.9). As we mentioned earlier, the boundary state $|B_*(\Psi)\rangle$ does not depend on the parameter s when the string field (1.6) satisfies the equation of motion (1.4). It is not difficult to see that the state (1.7) is s -independent only in the following three cases:

- $|B_*(\Psi)\rangle = |B\rangle$ for $x = y = 0$,
- $|B_*(\Psi)\rangle = 0$ for $x = y - 1 = \text{arbitrary}$,
- $|B_*(\Psi)\rangle = -|B\rangle$ for $x = -1, y = 1$.

We know that the boundary states for the first two cases are realized by solutions for the perturbative vacuum and the tachyon vacuum, respectively [46, 47]. However, we do not know what kind of solutions in open string field theory reproduce that for the third case. In [49], Okuda and Takayanagi introduced so-called ghost D-branes. They have negative tension and the corresponding boundary state is that for the perturbative vacuum with additional minus sign. Following the paper [49], we call solutions reproducing such a boundary state ghost brane solutions. Our result suggests that all the classical solutions in the KBc subalgebra can be classified into above three cases. In particular, there seems to exist no solution reproducing the boundary states for multiple D-brane backgrounds in the KBc subalgebra.⁶ This is the first main result of this paper.

In the above discussion, we classified possible boundary states in the KBc subalgebra by requiring s -independence as a necessary condition. In order to investigate concrete expression of classical solutions reproducing those boundary states, we consider two classes of formal solutions. We first consider the following class:

$$\Psi = U^{-1}QU = \sum_i I_i cKBcJ_i + \sum_i I_i cJ_i \frac{KB}{1 - \sum_j I_j J_j} \sum_k I_k cJ_k, \quad (1.10)$$

where the gauge parameter U is given by

$$U = 1 - \sum_i I_i cB J_i. \quad (1.11)$$

Note that when the indices i and j run over only one value, these formal solutions reproduce Okawa's formal solutions (1.2). Here, $I_i = I_i(K)$ and $J_i = J_i(K)$, and we sometimes omit the explicit indication of the argument K in the rest of this paper for simplicity. Applying our

⁶ We should mention that our formulae (1.7)-(1.9) is derived using the prescription developed in [51] based on the Schnabl gauge propagator. Although this prescription is well-established for solutions constructed from wedge-based states of finite width, it is not clear whether it is applicable for those containing wedge-based states of infinite width.

general formulae (1.7)-(1.9) of the boundary states to the string field (1.10), we obtain that

- $|B_*(\Psi)\rangle = |B\rangle$ for $\sum_j I_j(0)J_j(0) \neq 1$,
- $|B_*(\Psi)\rangle = 0$ for $\sum_j I_j(0)J_j(0) = 1$ and $\sum_j (I'_j(0)J_j(0) + I_j(0)J'_j(0)) \neq 0$.

The formal solution (1.10) does not satisfy our regularity conditions when the functions $I_i(K)$ and $J_i(K)$ satisfy $\sum_j I_j(0)J_j(0) = 1$ and $\sum_j (I'_j(0)J_j(0) + I_j(0)J'_j(0)) = 0$. In other words, this class of string fields satisfying our regularity conditions do not contain the ghost brane solution, and they reproduce the boundary states for the perturbative and the tachyon vacuum only. Note that our results seems to consistent with Erler's classification [24] of Okawa-type solutions (1.2) with respect to the energy calculation.

We next consider another class of formal solutions:

$$\Psi = Fc\frac{KB}{1-F^2}cF + Hc\frac{KB}{1-H^2}cH \quad \text{with} \quad FH = 1, \quad (1.12)$$

which cannot be written in the form (1.10). Applying our general formulae (1.7)-(1.9), we find that

- $|B_*(\Psi)\rangle = |B\rangle$ for $F(0) \neq 1$,
 - $|B_*(\Psi)\rangle = -|B\rangle$ for $F(0) = 1$ and $F'(0) \neq 0$.
- (1.13)

The formal solutions (1.12) does not satisfy our regularity conditions when the function $F(K)$ satisfies $F(0) = 1$ and $F'(0) = 0$. The first case reproduces the boundary state for the perturbative vacuum although it seems not to be gauge-equivalent to the perturbative vacuum as we mention later.⁷ The second case is nothing but the ghost brane solution. We propose a concrete candidate for the ghost brane solution in the following form:

$$\Psi_{ghost} = \sqrt{\frac{1-pK}{1-qK}}c\frac{(1-qK)}{p-q}Bc\sqrt{\frac{1-pK}{1-qK}} + \sqrt{\frac{1-qK}{1-pK}}c\frac{(1-pK)}{q-p}Bc\sqrt{\frac{1-qK}{1-pK}}, \quad (1.14)$$

where p and q are distinct positive constants. The state $\sqrt{(1-pK)/(1-qK)}$ is defined by the superposition of wedge states:

$$\sqrt{\frac{1-pK}{1-qK}} = \int_0^\infty dt_1 \int_0^\infty dt_2 \frac{e^{-t_1}}{\sqrt{\pi t_1}} \frac{e^{-t_2}}{\sqrt{\pi t_2}} e^{(qt_1+pt_2)K} (1-pK). \quad (1.15)$$

We carefully show that the ghost brane solution (1.14) satisfies the equation of motion and it has definite energy density of minus two times the D-brane tension.⁸ Although its physical

⁷ We showed that only the boundary states for the perturbative vacuum, the tachyon vacuum, and the ghost D-brane can be reproduced by classical solutions satisfying our regularity conditions. However, as the first case in (1.13) shows, it does not necessarily mean that solutions in our class are restricted to those for the three backgrounds.

⁸ The ghost D-brane discussed in [49] has the same energy density and reproduces the same boundary state as our ghost brane solution.

interpretation is still obscure, these results seem to be consistent with each other; we have no definite reason to rule out them. This is the second main result of this paper.

The organization of this paper is as follows. In section 2 we introduce the KBc subalgebra and our regularity conditions on string fields. In section 3 we evaluate the boundary states for general string fields in the KBc subalgebra under the regularity conditions introduced in section 2.2. After reviewing the construction by Kiermaier, Okawa, and Zwiebach [47], we derive the general formulae (1.7)-(1.9). By requiring s -independence as a necessary condition to satisfy the equation of motion, we show that possible solutions are only those reproducing the boundary states for the perturbative vacuum, the tachyon vacuum, and the ghost brane. In section 4 we consider the formal solutions (1.10). We show that this class of solutions include only those reproducing the boundary states for the tachyon vacuum and the perturbative vacuum. In section 5 we consider another class of formal solutions (1.12), and we propose a candidate for the ghost brane solution. Section 6 is devoted to discussion.

2 Setup

2.1 KBc subalgebra

The wedge state W_α with $\alpha \geq 0$ is defined by its BPZ inner product $\langle \varphi, W_\alpha \rangle$ as follows:

$$\langle \varphi, W_\alpha \rangle = \langle \mathcal{F} \circ \varphi(0) \rangle_{C_{\alpha+1}}. \quad (2.1)$$

Here and in what follows φ denotes a generic state in the Fock space and its corresponding operator is $\varphi(\xi)$ in the state-operator mapping. We denote the conformal transformation of $\varphi(\xi)$ under the map $\mathcal{F}(\xi)$ by $\mathcal{F} \circ \varphi(\xi)$, where

$$\mathcal{F}(\xi) = \frac{2}{\pi} \arctan \xi. \quad (2.2)$$

The coordinate z related through $z = \mathcal{F}(\xi)$ to the coordinate ξ on the upper half-plane used in the standard state-operator mapping is called the sliver frame. The correlation function is evaluated on the surface $C_{\alpha+1}$, which is the semi-infinite strip obtained from the upper half-plane of z by the identification $z \sim z + \alpha + 1$. We usually use the region $-1/2 \leq \Re z \leq 1/2 + \alpha$ for $C_{\alpha+1}$.

Just as the line integral L_0 of the energy-momentum tensor generates a surface e^{-tL_0} in the standard open string strip coordinate, the wedge state W_α can be thought of as being generated by a line integral of the energy-momentum tensor in the sliver frame. We denote the wedge state W_0 of zero width with an insertion of the line integral by K and we write the wedge state W_α as

$$W_\alpha = e^{\alpha K}. \quad (2.3)$$

An explicit definition of the state K is given by

$$\langle \varphi, K \rangle = \left\langle \mathcal{F} \circ \varphi(0) \int_{\frac{1}{2}+i\infty}^{\frac{1}{2}-i\infty} \frac{dz}{2\pi i} T(z) \right\rangle_{C_1}, \quad (2.4)$$

where $T(z)$ is the energy-momentum tensor and we use the doubling trick. Note that the line integral is from a boundary to the open string mid-point before using the doubling trick, while the line integral L_0 is from a boundary to the other boundary.

Just as the line integral L_0 is the BRST transformation of the line integral b_0 of the b ghost, the line integral that generates the wedge state is the BRST transformation of the same line integral with the energy-momentum tensor replaced by the b ghost. Correspondingly, we define the state B by

$$\langle \varphi, B \rangle = \left\langle \mathcal{F} \circ \varphi(0) \int_{\frac{1}{2}+i\infty}^{\frac{1}{2}-i\infty} \frac{dz}{2\pi i} b(z) \right\rangle_{C_1}. \quad (2.5)$$

By construction, the state K is the BRST transformation of B . Another important property of the state B is that $B^2 = 0$.

We also define the state c of ghost number 1 by a state based on the wedge state W_0 with a local insertion of $c(t)$ on the boundary. More explicitly, it is given by

$$\langle \varphi, c \rangle = \left\langle \mathcal{F} \circ \varphi(0) c\left(\frac{1}{2}\right) \right\rangle_{C_1}. \quad (2.6)$$

The state c satisfies [18]⁹

$$Qc = cKc, \quad c^2 = 0, \quad \{B, c\} = 1, \quad (2.7)$$

which follow from the BRST transformation of the c ghost, $Q \cdot c(t) = c\partial c(t)$, and the operator product expansions of the energy-momentum tensor, the b -ghost, and the c -ghost.

To summarize, the states K , B , and c satisfy the following algebraic relations called the KBc subalgebra:

$$B^2 = c^2 = 0, \quad \{B, c\} = 1, \quad QB = K, \quad QK = 0, \quad Qc = cKc, \quad [K, B] = 0. \quad (2.8)$$

As we mentioned in the introduction, we can define the states K , B , and c without specifying the D-brane configuration at the perturbative vacuum, and they always satisfy the algebraic relations (2.8).

⁹ Our definition of the states can be related to that in [18] as $K_{\text{here}} = (\pi/2) K_{\text{there}}$, $B_{\text{here}} = (\pi/2) B_{\text{there}}$, and $c_{\text{here}} = (2/\pi) c_{\text{there}}$.

2.2 Regularity conditions on string fields

In the calculation of boundary states, we require some regularity conditions on string fields. In this subsection we introduce those conditions and clarify our setup.

– **Regularity conditions** A state $F(K)$ defined by a superposition of wedge states is characterized by the following function $f(t)$:

$$F(K) = \int_0^\infty dt f(t) e^{tK}. \quad (2.9)$$

In the calculation of boundary states, we often encounter the expressions $F(0) = \int_0^\infty dt f(t)$ and $F'(0) = \int_0^\infty dt f(t) t$. We define two kinds of regularity conditions I and II by

$$\text{condition I : } \int_0^\infty dt f(t) \text{ is absolutely convergent,} \quad (2.10)$$

$$\text{condition II : } \int_0^\infty dt f(t) t \text{ is absolutely convergent.} \quad (2.11)$$

These conditions guarantee the finiteness of $F(0)$ and $F'(0)$, respectively.

– **Examples** Typical examples satisfying both of the two conditions are given by

$$\begin{aligned} e^{sK} &= \int_0^\infty dt \delta(t-s) e^{tK}, & K &= -\lim_{\epsilon \rightarrow 0} \int_0^\infty dt \delta'(t-\epsilon) e^{tK}, \\ \frac{1}{1-K} &= \int_0^\infty dt e^{-t} e^{tK}, & \frac{1}{\sqrt{1-K}} &= \int_0^\infty dt \frac{e^{-t}}{\sqrt{\pi t}} e^{tK}. \end{aligned} \quad (2.12)$$

When we define the sliver state Ω^∞ and the state $\sqrt{-K}$ by

$$\Omega^\infty = \lim_{\Lambda \rightarrow \infty} e^{\Lambda K} = \lim_{\Lambda \rightarrow \infty} \int_0^\infty dt \delta(t-\Lambda) e^{tK}, \quad (2.13)$$

$$\sqrt{-K} = -\lim_{\epsilon \rightarrow 0} \int_\epsilon^\infty dt \frac{1}{\sqrt{\pi t}} K e^{tK} = \lim_{\epsilon \rightarrow 0} \int_0^\infty dt \left(\frac{\delta(t-\epsilon)}{\sqrt{\pi t}} - \frac{1}{2\sqrt{\pi}} \theta(t-\epsilon) t^{-3/2} \right) e^{tK}, \quad (2.14)$$

they satisfy the condition I, but they do not satisfy the condition II. When we define the state $1/K$ by $1/K = \lim_{\epsilon \rightarrow 0} \frac{1}{K-\epsilon}$, it satisfies none of the two conditions:

$$\frac{1}{K} = -\lim_{\epsilon \rightarrow 0} \int_0^\infty dt e^{-\epsilon t} e^{tK}. \quad (2.15)$$

– **Regular string fields of ghost number one** Using the algebraic relations (2.8), any string field Ψ of ghost number 1 in the KBc subalgebra can be written as¹⁰

$$\Psi = \sum_i F_i(K) c B G_i(K) c H_i(K), \quad (2.16)$$

where $F_i(K)$, $G_i(K)$, and $H_i(K)$ are functions of K . The summation symbol \sum_i stands for sums and integrals over the labels of the string fields. In this paper, we consider string fields (2.16) with the states $F_i(K)$, $G_i(K)$, and $H_i(K)$ being superpositions of wedge states:

$$F_i(K) = \int_0^\infty dt f_i(t) e^{tK}, \quad G_i(K) = \int_0^\infty dt g_i(t) e^{tK}, \quad H_i(K) = \int_0^\infty dt h_i(t) e^{tK}. \quad (2.17)$$

We also assume that $F_i(K)$ and $H_i(K)$ satisfy both of the conditions I and II, and $G_i(K)$ satisfies the condition I.

These conditions are all we need to assume when we calculate boundary states. With these conditions, we can safely calculate the boundary state for Ψ . Note that our regularity conditions do not exclude identity-based string fields, which are usually thought to be singular with respect to energy calculation. Accordingly, our calculation of boundary states can be also applicable to identity-based solutions. In the next section we evaluate the boundary state for (2.16) under these assumptions.

3 Boundary states in KBc subalgebra

In this section we discuss possible boundary states which can be reproduced from general string fields in the KBc subalgebra under the regularity conditions introduced in section 2.2. After reviewing the construction of boundary states in [47], we derive the general formulae (1.7)-(1.9). By requiring the s -independence as a necessary condition to satisfy the equation of motion, we show that possible classical solutions are only those reproducing the boundary states for the perturbative vacuum, the tachyon vacuum, and the ghost brane.

3.1 Review

In this subsection we briefly review the construction of boundary states proposed by Kiermaier, Okawa, and Zwiebach (KOZ) [47]. Since it is slightly complicated, we start from the basic strategy of the construction.

¹⁰ To be rigorous, the use of the algebraic relation $\{B, c\} = 1$ can affect regularity of string fields in general. For example, we can rewrite a string field Ψ in the form of $\Psi = e^{K/2} c B \frac{1}{K} c B K c e^{K/2}$ as $\Psi = e^{K/2} c B c e^{K/2}$, formally using the relation $\{B, c\} = 1$ and $\frac{1}{K} K = 1$. Here, the regularity of Ψ seems to be changed by the use of $\{B, c\} = 1$. In this paper, we assume that functions of K in the string fields are defined by superpositions of wedge states and they satisfy the regularity condition I. Under this assumption, we can use the relation $\{B, c\} = 1$ without subtlety and any string field of ghost number 1 can be written in the form of (2.16).

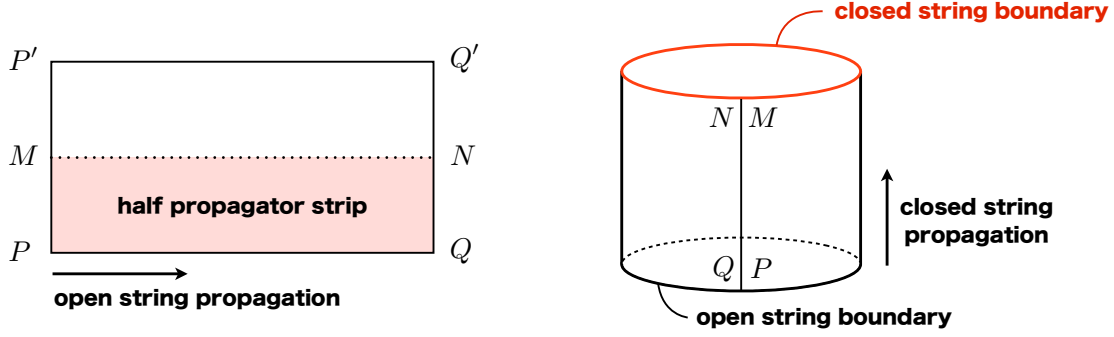


Figure 1: The rectangular $PQQ'P'$ is the open string world-sheet strip $e^{-s\mathcal{L}}$, and the dotted line MN is the midpoint propagation of the open string. The rectangular $PQNM$ describes the half propagator strip. Identifying MP and NQ , we obtain a cylinder in the right figure.

– **Basic strategy** We first construct the boundary state $|B\rangle$ for the boundary conformal field theory (BCFT) corresponding to the perturbative vacuum. Consider a open string world-sheet strip $e^{-s\mathcal{L}}$ associated with the propagator of the open string. In the linear b gauge, the generator \mathcal{L} of the world-sheet strip is given by $\mathcal{L} = \{Q, \mathcal{B}_{\text{lin}}\}$, where \mathcal{B}_{lin} denotes the ghost number -1 operator, which determines the gauge condition:

$$\mathcal{B}_{\text{lin}}\Psi = 0 \quad \text{with} \quad \mathcal{B}_{\text{lin}} = \oint \frac{d\xi}{2\pi i} v(\xi) b(\xi). \quad (3.1)$$

We then cut the strip along the trajectory of the midpoint propagation of the open string. We call one of the resulting pieces a *half propagator strip* (see Figure 1). By identifying the two edges MP and NQ , which represent the initial and final half-string states, we obtain a cylinder. One of two boundaries of the cylinder is the open string boundary, where the boundary conditions are defined by the original BCFT, and the other is the trajectory of the midpoint propagation, which we call a closed string boundary. Path integrals over the cylinder define a closed string state at the closed string boundary. Then, this closed string state reproduces the boundary state $|B\rangle$ of the original BCFT after an appropriate exponential action of $L_0 + \tilde{L}_0$, which generates the closed string propagation. Note that we can arbitrarily choose a gauge condition and a parameter s representing the open string propagation in the construction.

Next, let us consider the boundary state $|B_*\rangle$ for the BCFT_* associated with a classical solution Ψ . The open string propagation around the background Ψ is generated by $\mathcal{L}_* = \{Q_*, \mathcal{B}_{\text{lin}}\}$, where Q_* is the kinetic operator around the background Ψ . Then, it is expected that the boundary state $|B_*\rangle$ for BCFT_* can be obtained by replacing \mathcal{L} with \mathcal{L}_* in the previous construction. In [47], it was discussed that the closed string state $|B_*(\Psi)\rangle$ constructed in this way coincides with the boundary state $|B_*\rangle$ up to some BRST-exact terms. That's the basic strategy of the KOZ construction.

– **Properties** Before going to the concrete description of the construction, we mention some properties of the closed string state $|B_*(\Psi)\rangle$ of the KOZ construction. In [47], it was shown that $|B_*(\Psi)\rangle$ satisfies the following three properties, which can be considered as consistency requirements for its interpretation as a boundary state:

$$\mathcal{Q}|B_*(\Psi)\rangle = 0, \quad (b_0 - \tilde{b}_0)|B_*(\Psi)\rangle = 0, \quad (L_0 - \tilde{L}_0)|B_*(\Psi)\rangle = 0, \quad (3.2)$$

where \mathcal{Q} is the BRST operator of the closed bosonic string. When two classical solutions are gauge-equivalent, corresponding closed string states $|B_*(\Psi)\rangle$ are expected to be physically equivalent. Indeed, $|B_*(\Psi)\rangle$ is invariant under the gauge transformation δ_χ of the classical solution Ψ up to \mathcal{Q} -exact terms:

$$\delta_\chi |B_*(\Psi)\rangle = \mathcal{Q}\text{-exact}. \quad (3.3)$$

In the construction, we can choose any regular gauge condition of the open string propagator and the parameter s associated with the propagation length. It is expected that physical properties of the state $|B_*(\Psi)\rangle$ do not depend on these choices. Indeed, it is invariant up to \mathcal{Q} -exact terms under a variation $\delta\mathcal{B}_{\text{lin}}$ of the gauge condition and a variation of the parameter s :

$$\delta\mathcal{B}_{\text{lin}} |B_*(\Psi)\rangle = \mathcal{Q}\text{-exact}, \quad \partial_s |B_*(\Psi)\rangle = \mathcal{Q}\text{-exact}. \quad (3.4)$$

We emphasize that the equation of motion for Ψ was used to derive all the above properties.

We also note that the calculation of $|B_*(\Psi)\rangle$ in the limit $s \rightarrow 0$ can be directly related to that of the gauge invariant observables [41, 42], which are conjectured [43] to represent the difference between on-shell closed string one-point functions on the disk for BCFT and BCFT $_*$.

3.1.1 Construction

Then, we move on to the concrete construction. As we mentioned above, the open string propagator in the linear b gauge is generated by

$$\mathcal{L} = \{Q, \mathcal{B}_{\text{lin}}\} \quad \text{with} \quad \mathcal{B}_{\text{lin}} = \oint \frac{d\xi}{2\pi i} v(\xi) b(\xi) = \oint \frac{d\xi}{2\pi i} \frac{\mathcal{F}_{\text{lin}}(\xi)}{\mathcal{F}'_{\text{lin}}(\xi)} b(\xi), \quad (3.5)$$

where we introduced a function $\mathcal{F}_{\text{lin}}(\xi)$ given by $v(\xi) = \frac{\mathcal{F}_{\text{lin}}(\xi)}{\mathcal{F}'_{\text{lin}}(\xi)}$ and $\mathcal{F}_{\text{lin}}(1) = 1$. It is useful to define two coordinates w and z by

$$e^w = 2z = \mathcal{F}_{\text{lin}}(\xi). \quad (3.6)$$

In these frames, the operator \mathcal{L} generates a translation along the real axis and a dilatation, respectively, for the w -frame and the z -frame. To define the half propagator strip, we use the

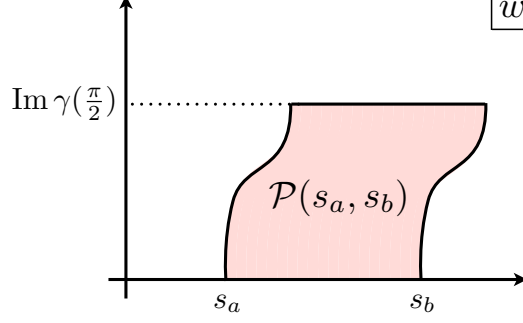


Figure 2: A half propagator strip $\mathcal{P}(s_a, s_b)$. The edge on the real axis is the open string boundary, and that on $\text{Im } w = \text{Im } \gamma(\frac{\pi}{2})$ is the trajectory of the open string midpoint.

operator $\mathcal{L}_R(t)$ defined in the w -frame by

$$\mathcal{L}_R(t) \equiv \int_t^{\gamma(\frac{\pi}{2})+t} \left[\frac{dw}{2\pi i} T(w) + \frac{d\bar{w}}{2\pi i} \tilde{T}(\bar{w}) \right], \quad (3.7)$$

where t is a real number and the integral over w is along the curve $\gamma(\theta)$ ($0 \leq \theta \leq \pi$):

$$\gamma(\theta) = w(\xi = e^{i\theta}). \quad (3.8)$$

We then define the half propagator strip in w frame as follows:

$$\mathcal{P}(s_a, s_b) = \text{P exp} \left[- \int_{s_a}^{s_b} dt \mathcal{L}_R(t) \right], \quad (3.9)$$

where P exp denotes the path-ordered exponential and the parameters s_a and s_b describe positions of the strip on the real axis (see Figure 2).

Using the half propagator strip, we construct the boundary state $|B\rangle$ for the original background as follows:

$$|B\rangle = e^{\frac{\pi^2}{s}(L_0 + \tilde{L}_0)} \oint_s \mathcal{P}(0, s), \quad (3.10)$$

where the operation \oint_s denotes an identification of the left and right boundaries of the surface $\mathcal{P}(0, s)$, which is realized as $w \sim w + s$ in the w -frame and $z \sim e^s z$ in the z -frame. The operator $e^{\frac{\pi^2}{s}(L_0 + \tilde{L}_0)}$ generates a propagation of the closed string from the closed string boundary to the open string boundary.

Next, we construct the closed string state $|B_*(\Psi)\rangle$ by replacing \mathcal{L} with \mathcal{L}_* . The operator $\mathcal{L}_R(t)$ is then replaced by $\mathcal{L}_R(t) + \{\mathcal{B}_R(t), \Psi\}$,¹¹ and the half propagator strip is also replaced

¹¹ It is because the action of the operator $\{Q_*, \mathcal{B}_{\text{lin}}\}$ on a string field A is given by

$$\{Q_*, \mathcal{B}_{\text{lin}}\}A = [\mathcal{L}_R A + (-)^A (\mathcal{B}_R A) \Psi - (-)^A \mathcal{B}_R (A \Psi)] + [\mathcal{L}_L A + \Psi (\mathcal{B}_L A) + \mathcal{B}_L (\Psi A)],$$

where $\mathcal{L}_{L/R}$ and $\mathcal{B}_{L/R}$ represent the left/right half of the line integrals \mathcal{L} and \mathcal{B}_{lin} .

as

$$\mathcal{P}(s_a, s_b) \rightarrow \mathcal{P}_*(s_a, s_b) \equiv \text{P exp} \left[- \int_{s_a}^{s_b} dt (\mathcal{L}_R(t) + \{\mathcal{B}_R(t), \Psi\}) \right]. \quad (3.11)$$

Using this deformed half propagator strip $\mathcal{P}_*(s_a, s_b)$, we construct the closed string state $|B_*(\Psi)\rangle$ around the new background in analogy with $|B\rangle$:

$$|B_*(\Psi)\rangle = e^{\frac{\pi^2}{s}(L_0 + \tilde{L}_0)} \oint_s \mathcal{P}_*(0, s). \quad (3.12)$$

Expanding the path-ordered exponential in $\mathcal{P}_*(0, s)$, we can write the closed string state $|B_*(\Psi)\rangle$ as a power series in the classical solution Ψ :

$$\begin{aligned} & |B_*(\Psi)\rangle \\ &= \sum_{k=0}^{\infty} |B_*^{(k)}(\Psi)\rangle \\ &= \sum_{k=0}^{\infty} (-)^k e^{\frac{\pi^2}{s}(L_0 + \tilde{L}_0)} \oint_s \int_0^s ds_1 \cdots \int_{s_{i-1}}^s ds_i \cdots \int_{s_{k-1}}^s ds_k \mathcal{P}(0, s_1) \{\mathcal{B}_R(s_1), \Psi\} \mathcal{P}(s_1, s_2) \\ & \quad \times \cdots \mathcal{P}(s_{i-1}, s_i) \{\mathcal{B}_R(s_i), \Psi\} \mathcal{P}(s_i, s_{i+1}) \cdots \mathcal{P}(s_{k-1}, s_k) \{\mathcal{B}_R(s_k), \Psi\} \mathcal{P}(s_k, s), \end{aligned} \quad (3.13)$$

where $|B_*^{(0)}(\Psi)\rangle = |B\rangle$. Products of half propagators and solutions are defined by some gluing conditions, which we explain in the next subsection for half propagators in the Schnabl gauge. In the following, we use this expression to construct boundary states from classical solutions.

3.1.2 Schnabl gauge calculation

In general, it is difficult to construct boundary states from wedge-based classical solutions because of the complicated gluing operation of half propagator strips and classical solutions. However, it becomes tractable when we use half propagator strips associated with Schnabl gauge propagators.¹²

In the Schnabl gauge calculation, it is useful to perform in the z -frame, which coincides with the sliver frame $z = \frac{2}{\pi} \arctan \xi$. In this frame, the half propagator strip $\mathcal{P}(s_{i-1}, s_i)$ can be described as a surface in the region

$$\frac{1}{2} e^{s_{i-1}} \leq \text{Re } z_i \leq \frac{1}{2} e^{s_i}. \quad (3.14)$$

¹² The Schnabl gauge is singular in the sense that the propagator does not generate midpoint propagation. Therefore, it should be understood as a singular limit of one-parameter family of regular linear b gauges [50] when the midpoint propagation plays an important role. In the original paper [47], the Schnabl gauge calculation of boundary states is introduced in this way, and it is justified for classical solutions constructed from wedge-based states of finite width [51]. Although it is not clear whether it is justified when classical solutions contain wedge-based states of infinite width, we expect that the use of the Schnabl gauge calculation is justified when our regularity conditions are satisfied.

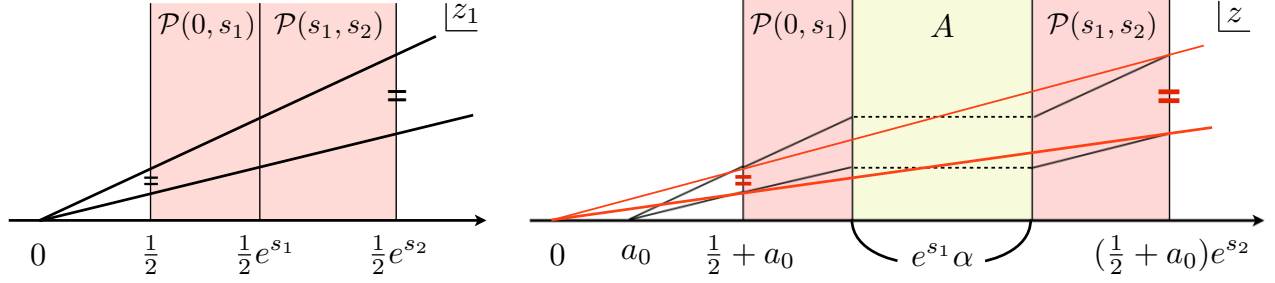


Figure 3: The operation \mathcal{f}_{s_2} on the half propagator $\mathcal{P}(0, s_1)\mathcal{P}(s_1, s_2)$ is realized by the identification $z_1 \sim e^{s_2} z_1$ (the left figure). After inserting a string field A , the operation \mathcal{f}_{s_2} on $\mathcal{P}(0, s_1) A \mathcal{P}(s_1, s_2)$ is complicatedly realized in the z_1 frame. However, in the natural z frame, it is naturally realized by $z \sim e^{s_2} z$ (the right figure).

Then, let us consider an insertion of a wedge-based state A of width α between two half propagator strips $\mathcal{P}(0, s_1)$ and $\mathcal{P}(s_1, s_2)$. Associated with the fact that the open string propagator generates dilatation in the z -frame, the gluing condition of $\mathcal{P}(0, s_1)$ and A is given by

$$z_1 = e^{s_1} z_A, \quad (3.15)$$

where z_A is the coordinate of A in the sliver frame. In the same way, the gluing condition of A and $\mathcal{P}(s_1, s_2)$ is given by

$$e^{s_1}(z_A - \alpha) = z_2. \quad (3.16)$$

When we describe the surface $\mathcal{P}(0, s_1) A \mathcal{P}(s_1, s_2)$ in the z_1 -frame, it is located in the region

$$\frac{1}{2} \leq \text{Re } z_1 \leq e^{s_1} \alpha + \frac{1}{2} e^{s_2}. \quad (3.17)$$

To restore the fact that $\mathcal{P}(0, s_1)\mathcal{P}(s_1, s_2) = \mathcal{P}(0, s_2)$ generates a dilatation $z \rightarrow e^{s_2} z$, it is useful to introduce the following natural z -frame (see Figure 3):

$$z = z_1 + a_0 \quad \text{with} \quad a_0 = \frac{e^{s_1}}{e^{s_2} - 1} \alpha. \quad (3.18)$$

In this frame, the surface is located in the region

$$\frac{1}{2} + a_0 \leq \text{Re } z \leq e^{s_2} \left(\frac{1}{2} + a_0 \right), \quad (3.19)$$

and the operation \mathcal{f}_{s_2} on $\mathcal{P}(0, s_1) A \mathcal{P}(s_1, s_2)$ can be realized by the identification $z \sim e^{s_2} z$.

Above discussion can be generalized straightforwardly to the case with multi insertions of wedge-based states:

$$\mathcal{P}(0, s_1) A_1 \mathcal{P}(s_1, s_2) A_2 \mathcal{P}(s_2, s_3) \dots \mathcal{P}(s_k - 1, s_k) A_k \mathcal{P}(s_k, s), \quad (3.20)$$

where A_i is a wedge-based state of width α_i . In the z_1 -frame, it is located in the region

$$\frac{1}{2} \leq \text{Re } z_1 \leq \sum_{j=1}^k e^j \alpha_j + \frac{1}{2} e^s, \quad (3.21)$$

and we define the natural z -frame by

$$z = z_1 + a_0 \quad \text{with} \quad a_0 = \frac{1}{e^s - 1} \sum_{j=1}^k e^{sj} \alpha_j. \quad (3.22)$$

In the natural z -frame, the surface is located in the region

$$\frac{1}{2} + a_0 \leq \text{Re } z \leq e^{s_2} \left(\frac{1}{2} + a_0 \right), \quad (3.23)$$

and the operation \oint_s can be realized by the identification $z \sim e^s z$. We also note that the conformal mapping between the coordinate z of the natural z -frame and the coordinate z_{A_i} of A_i in the sliver frame is given by

$$z = \ell_i + e^{s_i} z_{A_i} \quad \text{with} \quad \ell_i = \sum_{j=1}^{i-1} \alpha_j e^{s_j} + a_0 \quad \text{and} \quad \ell_1 = a_0. \quad (3.24)$$

Then, let us apply the above discussion to the boundary state (3.13). When the classical solution Ψ is constructed from wedge-based states, $\{\mathcal{B}_R(s_i), \Psi\}$ in (3.13) can be written as a sum of states in the form of $\{\mathcal{B}_R(s_i), A_i\}$, where A_i is a wedge-based state of width α_i . In the natural z frame, this commutator is expressed as

$$-\{\mathcal{B}_R(s_i), A_{\alpha_i}\} \rightarrow \oint \frac{dz}{2\pi i} (z - \ell_i) b(z) [\dots] - e^{s_i} \alpha_i [\dots] \mathcal{B}, \quad (3.25)$$

where $[\dots]$ represents the operator insertions coming from A_i and \mathcal{B} is a line integral defined by

$$\mathcal{B} = \int_{i\infty}^{-i\infty} \frac{dz}{2\pi i} b(z). \quad (3.26)$$

Then, the calculation of boundary states is reduced to the evaluation of quantities in the form of $\mathcal{B}[\dots]$. Since the gluing operation \oint_s leads to the identification $z \sim e^s z$ in the natural z -frame, we obtain

$$\mathcal{B}[\dots] = e^s (-1)^{[\dots]} [\dots] \mathcal{B}, \quad (3.27)$$

where $[\cdots]$ represents all the insertions of operators on the surface. Therefore, we conclude that

$$\mathcal{B}[\cdots] = \frac{e^s}{e^s - 1} \oint \frac{dz}{2\pi i} b(z) [\cdots], \quad (3.28)$$

where the contour encircles all the operator insertions $[\cdots]$ counterclockwise. In the following section, we use the formulae (3.25) and (3.28) to calculate boundary states in the KBc subalgebra.

3.2 Boundary states from string fields in KBc subalgebra

In this subsection we calculate the closed string state $|B_*(\Psi)\rangle$ defined by (3.13) for general string fields Ψ in the KBc subalgebra:

$$\Psi = \sum_i F_i(K) c B G_i(K) c H_i(K). \quad (3.29)$$

As we mentioned in section 2, we assume that $F_i(K)$, $G_i(K)$, and $H_i(K)$ are superpositions of wedge states, and they satisfy the regularity conditions introduced in section 2.2. We note that the closed string state $|B_*(\Psi)\rangle$ is well-defined, although it does not satisfy some properties of boundary states unless Ψ satisfies the equation of motion.

We first rewrite $\{\mathcal{B}_R(s_i), \Psi\}$ in (3.13) using the formula (3.25). Using the expression of Ψ as a superposition of wedge-based states,

$$\Psi = \sum_i \int_0^\infty d\alpha_i f_i(\alpha_i) \int_0^\infty d\beta_i g_i(\beta_i) \int_0^\infty d\gamma_i h_i(\gamma_i) e^{\alpha_i K} c B e^{\beta_i K} c e^{\gamma_i K}, \quad (3.30)$$

the operator insertions in the natural z -frame coming from each integrand $e^{\alpha_i K} c B e^{\beta_i K} c e^{\gamma_i K}$ are given by

$$e^{-s_i} c((\frac{1}{2} + \alpha_i) e^{s_i} + \ell_i) \mathcal{B} c((\frac{1}{2} + \alpha_i + \beta_i) e^{s_i} + \ell_i). \quad (3.31)$$

Using the formula (3.25), the operator insertions coming from $-\{\mathcal{B}_R(s_j), e^{\alpha_i K} c B e^{\beta_i K} c e^{\gamma_i K}\}$ are

$$\left(\frac{1}{2} + \alpha_i\right) \mathcal{B} c((\frac{1}{2} + \alpha_i + \beta_i) e^{s_i} + \ell_i) + \left(\frac{1}{2} - \gamma_i\right) c((\frac{1}{2} + \alpha_i) e^{s_j} + \ell_j) \mathcal{B}. \quad (3.32)$$

We notice that the first term and the second term in (3.32) are schematically in the form of $\mathcal{B}c$ and $c\mathcal{B}$, respectively.

We then consider the operator insertions in $|B_*^{(k)}(\Psi)\rangle$ defined by (3.13). Let us start from qualitative discussion. As we mentioned above, we have $\mathcal{B}c$ -type and $c\mathcal{B}$ -type insertions from

each $\{\mathcal{B}_R(s_i), \Psi\}$. Since $\mathcal{B}^2 = 0$, cross terms between these two types vanish. Then, we have two types of operator insertions schematically in the form of

$$\prod_{i=1}^k \left(\mathcal{B} c(t_i) \right) \quad \text{and} \quad \prod_{i=1}^k \left(c(t_i) \mathcal{B} \right). \quad (3.33)$$

Furthermore, using $\{\mathcal{B}, c(t_i)\} = 1$ and the formula (3.28), they can be written as

$$\prod_{i=1}^k \left(\mathcal{B} c(t_i) \right) = \mathcal{B} c(t_k) = \frac{e^s}{e^s - 1} \quad \text{and} \quad \prod_{i=1}^k \left(c(t_i) \mathcal{B} \right) = c(t_1) \mathcal{B} = -\frac{1}{e^s - 1}, \quad (3.34)$$

where we notice that (3.34) does not depend on the insertion points of c ghosts. Therefore, we can ignore the argument of c ghosts, and $-\{\mathcal{B}_R(s_i), \Psi\}$ can be written as follows:

$$\begin{aligned} & -\{\mathcal{B}_R(s_i), \Psi\} \\ & \rightarrow \sum_i \int_0^\infty d\alpha_i f_i(\alpha_i) \int_0^\infty d\beta_i g_i(\beta_i) \int_0^\infty d\gamma_i h_i(\gamma_i) \left[\left(\frac{1}{2} + \alpha_i \right) \mathcal{B} c + \left(\frac{1}{2} - \gamma_i \right) c \mathcal{B} \right] \\ & = x \mathcal{B} c + y c \mathcal{B}, \end{aligned} \quad (3.35)$$

where x and y are c -numbers given by

$$x = \sum_i G_i(0) \left(\frac{1}{2} F_i(0) H_i(0) + F'_i(0) H_i(0) \right), \quad (3.36)$$

$$y = \sum_i G_i(0) \left(\frac{1}{2} F_i(0) H_i(0) - F'_i(0) H_i(0) \right). \quad (3.37)$$

Here the regularity conditions on string fields introduced in section 2.2 guarantee the finiteness of x and y . In the same way, the operator insertions in $|B_*^{(k)}(\Psi)\rangle$ can be written as

$$(-)^k \prod_{i=1}^k \{\mathcal{B}_R(s_i), \Psi\} \rightarrow (x \mathcal{B} c)^k + (y c \mathcal{B})^k = x^k \mathcal{B} c + y^k c \mathcal{B} = \frac{e^s}{e^s - 1} x^k - \frac{1}{e^s - 1} y^k. \quad (3.38)$$

We then obtain

$$|B_*^{(k)}(\Psi)\rangle = \frac{s^k}{k!} \left(\frac{e^s}{e^s - 1} x^k - \frac{1}{e^s - 1} y^k \right) |B\rangle, \quad (3.39)$$

where the factor $s^k/k!$ comes from

$$\int_0^s ds_1 \int_{s_1}^s ds_2 \cdots \int_{s_{k-1}}^s ds_k = \frac{s^k}{k!}. \quad (3.40)$$

Therefore, the all order state $|B_*(\Psi)\rangle$ is¹³

$$|B_*(\Psi)\rangle = \sum_{k=0}^{\infty} |B_*^{(k)}(\Psi)\rangle = \frac{e^{(x+1)s} - e^{ys}}{e^s - 1} |B\rangle. \quad (3.41)$$

¹³ When the index i in (3.29) runs over one value, our result reproduces that in the previous paper [46].

We emphasize that the closed string state $|B_*(\Psi)\rangle$ is proportional to the boundary state $|B\rangle$ for the original BCFT, and the information about the sting field Ψ only appears in the c -numbers x and y .

3.3 The s -dependence of boundary states

As we mentioned earlier, the closed string state $|B_*(\Psi)\rangle$ does not depend on the parameter s when Ψ satisfies the equation of motion. In this subsection we explore which class of classical solutions can be in the KBc subalgebra by requiring the s -independence of $|B_*(\Psi)\rangle$ as a necessary condition to satisfy the equation of motion.

By differentiating (3.41) with respect to s , we obtain that

$$\frac{\partial}{\partial s}|B_*(\Psi)\rangle = \frac{1}{(e^s - 1)^2} [xe^{(2+x)s} - (1+x)e^{(1+x)s} + (1-y)e^{(1+y)s} + ye^{ys}] |B\rangle. \quad (3.42)$$

We first notice that the s -dependence does not vanish when none of the following conditions are satisfied: $y = x$, $y = x + 1$, and $y = x + 2$. Then, let us consider the following three cases:

1. $y = x$

The s -derivative (3.42) is given by

$$\frac{\partial}{\partial s}|B_*(\Psi)\rangle = \frac{1}{(e^s - 1)^2} [xe^{(2+x)s} - 2xe^{(1+x)s} + xe^{xs}] |B\rangle, \quad (3.43)$$

and it vanishes if and only if $x = 0$. For $x = y = 0$, the closed string state (3.41) takes the form

$$|B_*(\Psi)\rangle = |B\rangle, \quad (3.44)$$

and the boundary state for the perturbative vacuum can be reproduced by this class of string fields.

2. $y = x + 1$

The s -derivative (3.42) is identically zero:

$$\frac{\partial}{\partial s}|B_*(\Psi)\rangle = \frac{1}{(e^s - 1)^2} [xe^{(2+x)s} - (1+x)e^{(1+x)s} - xe^{(2+x)s} + (1+x)e^{(1+x)s}] |B\rangle = 0. \quad (3.45)$$

For $y = x + 1 = \text{arbitrary}$, the closed string state (3.41) takes the form

$$|B_*(\Psi)\rangle = 0, \quad (3.46)$$

and the boundary state for the tachyon vacuum can be reproduced by this class of string fields.

3. $y = x + 2$

The s -derivative (3.42) is given by

$$\frac{\partial}{\partial s} |B_*(\Psi)\rangle = \frac{1}{(e^s - 1)^2} [2(1+x)e^{(2+x)s} - (1+x)e^{(1+x)s} - (1+x)e^{(3+x)s}] |B\rangle, \quad (3.47)$$

and it vanishes if and only if $x = -1$. For $x = -1$ and $y = 1$, the closed string state (3.41) takes the form

$$|B_*(\Psi)\rangle = -|B\rangle. \quad (3.48)$$

This type of string fields reproduce boundary states for the original BCFT with -1 factor, which coincide with that for the ghost D-brane introduced in [49]. We call solutions reproducing such a boundary state ghost brane solutions. In section 5, we give an example of this type of classical solutions.

Summarizing above discussions, the s -dependence of the closed string state $|B_*(\Psi)\rangle$ vanishes only in the following three cases: $x = y = 0$, $x = y - 1 = \text{arbitrary}$, and $x = y - 2 = -1$. They reproduce the boundary states for the perturbative vacuum, the tachyon vacuum, and the ghost D-brane, respectively.

– **Comments on singular string fields and their regularization** Our results suggest that we are not able to construct multiple D-brane solutions in the KBc subalgebra without relaxing our regularity conditions. Indeed, the proposed multiple D-brane solutions in [36, 39] contain singular expression such as $1/K$, and it is not known how to regularize them consistently. In the last of this section we comment on regularization of singular string fields.

Suppose that a string field Ψ of ghost number 1 contains a singular expression such as $1/K$, and we regularize it using a one-parameter family Ψ_ϵ of regular string fields satisfying our regularity conditions:

$$\Psi = \lim_{\epsilon \rightarrow 0} \Psi_\epsilon \quad \text{with} \quad \Psi_\epsilon = \sum_i F_i^\epsilon(K) c B G_i^\epsilon(K) c H_i^\epsilon(K). \quad (3.49)$$

For Ψ_ϵ ($\epsilon \neq 0$), we can evaluate the closed string state $|B_*(\Psi)\rangle$ using the general formula (3.41):

$$|B_*(\Psi_\epsilon)\rangle = \left[\frac{e^s}{e^s - 1} \exp(x_\epsilon s) - \frac{1}{e^s - 1} \exp(y_\epsilon s) \right] |B\rangle, \quad (3.50)$$

where x_ϵ and y_ϵ are the parameters (3.36) and (3.37) for Ψ_ϵ . We notice that the s -independence of boundary states prohibits multiple D-brane solutions as long as we formally take the limit $\epsilon \rightarrow 0$ in the expression (3.50). We should mention, however, that we used the Schnabl gauge calculation to derive our formula, and it is not clear whether the Schnabl gauge calculation is justified in the limit $\epsilon \rightarrow 0$. In section 6 we discuss what we should do to clarify this point.

4 Constraints on pure-gauge ansatz

In this section we apply the results in the previous section to string fields formally in the pure-gauge form. We show that this class of solutions include only those reproducing the following boundary states: $|B_*\rangle = |B\rangle$ and 0.

4.1 Pure-gauge ansatz in KBc subalgebra

Any string field U of ghost number 0 in the KBc subalgebra can be written as

$$U = H(K) - \sum_i I_i(K) c B J_i(K), \quad (4.1)$$

where $H(K)$, $I_i(K)$, and $J_i(K)$ are arbitrary functions of the state K . We formally define the inverse of U by

$$U^{-1} = \left[1 - H^{-1}(K) \sum_j I_j(K) B c J_j(K) \right] \frac{1}{H(K) - \sum_i I_i(K) J_i(K)}, \quad (4.2)$$

where we assumed that $H(K) \neq 0$ and $H(K) - \sum_i I_i(K) J_i(K) \neq 0$. Then, the most general form of the pure-gauge ansatz in the KBc subalgebra is given by

$$\begin{aligned} \Psi &= U^{-1} Q U \\ &= \sum_i H^{-1} I_i c K B c J_i + \sum_i H^{-1} I_i c J_i \frac{K B}{H - \sum_j I_j J_j} \sum_k I_k c J_k. \end{aligned} \quad (4.3)$$

When $H(K) \neq 0$, we can set $H(K) = 1$ without loss of generality, and therefore, we consider the following class of formal solutions in the rest of this section:

$$\Psi = \sum_i I_i c K B c J_i + \sum_{i,j} I_i c B G_{ij} c J_j \quad \text{with} \quad G_{ij} = \frac{K J_i I_j}{1 - \sum_k I_k J_k}. \quad (4.4)$$

Note that when the indices i and j in (4.4) run over only one value, we obtain Okawa's formal solutions (1.2). In the following, we assume that $I_i(K)$ and $J_i(K)$ satisfy the regularity conditions I and II introduced in section 2.2, and $G_{ij}(K)$ satisfies the regularity condition I. We do not assume that U and U^{-1} are regular: the formal solution Ψ can be regular even if U or U^{-1} is singular as is the case of Schnabl's original solution [2, 18] and the simple solution [17] for the tachyon vacuum.

4.2 Constraints using boundary states

We then clarify which class of solutions are in this class of formal solutions (4.3) using the property of boundary states. Applying the general formulae, we obtain that

$$x = \sum_{ij} \left[G_{ij}(0) \left(\frac{1}{2} I_i(0) J_j(0) + I'_i(0) J_j(0) \right) \right], \quad (4.5)$$

$$y = \sum_{ij} \left[G_{ij}(0) \left(\frac{1}{2} I_i(0) J_j(0) - I_i(0) J'_j(0) \right) \right], \quad (4.6)$$

where the parameters x and y are well-defined because of the regularity conditions. Then, let us consider the following three cases.

1. $\sum_j I_j(0) J_j(0) \neq 1$

The regularity conditions imply

$$G_{ij}(0) = \frac{0 \cdot J_i(0) I_j(0)}{1 - \sum_k I_k(0) J_k(0)} = 0, \quad (4.7)$$

which leads to $x = y = 0$. Therefore, the boundary state for the perturbative vacuum can be reproduced by this class of string fields: $|B_*(\Psi)\rangle = |B\rangle$.

2. $\sum_j I_j(0) J_j(0) = 1$ and $\sum_j (I'_j(0) J_j(0) + I_j(0) J'_j(0)) \neq 0$

The regularity conditions imply

$$\sum_k I_k(z) J_k(z) = 1 + \sum_k (I'_k(0) J_k(0) + I_k(0) J'_k(0)) z + \mathcal{O}(z^2), \quad (4.8)$$

which leads to

$$G_{ij}(0) = - \frac{J_i(0) I_j(0)}{\sum_k [I'_k(0) J_k(0) + I_k(0) J'_k(0)]}. \quad (4.9)$$

We then obtain

$$x = - \frac{\frac{1}{2} + \sum_i I'_i(0) J_i(0)}{\sum_k [I'_k(0) J_k(0) + I_k(0) J'_k(0)]}, \quad (4.10)$$

$$y = - \frac{\frac{1}{2} - \sum_i I_i(0) J'_i(0)}{\sum_k [I'_k(0) J_k(0) + I_k(0) J'_k(0)]} = x + 1. \quad (4.11)$$

Therefore, the boundary state for the tachyon vacuum can be reproduced by this class of string fields: $|B_*(\Psi)\rangle = 0$.

3. $\sum_j I_j(0) J_j(0) = 1$ and $\sum_j (I'_j(0) J_j(0) + I_j(0) J'_j(0)) = 0$

The regularity condition on $G_{ij}(K)$ implies $I_j(0) J_i(0) = 0$ for any i and j . However, it is incompatible with the assumption $\sum_j I_j(0) J_j(0) = 1$. Therefore, no string fields in this class satisfy our regularity conditions.

Summarizing above discussions, the boundary states for the perturbative and the tachyon vacuum can be reproduced by the formal solutions based on the pure-gauge ansatz in the KBc subalgebra (4.3). In particular, the ghost brane solutions are not in this class of formal solutions.

4.3 Examples

In this subsection we consider two examples for the pure-gauge ansatz. We first consider the solutions for tachyon condensation [2, 17], whose boundary states are already evaluated in [47, 46]. We make an observation that only the so-called phantom term contributes to the boundary states for solutions in the KBc subalgebra. We then consider the proposed multiple D-brane solutions [36, 39]. After formally applying our results to the proposed solutions, we discuss two types of regularization [39, 44] and [45] in our framework.

4.3.1 Solutions for tachyon condensation and the phantom term

– **Schnabl’s original solution** We start from Schnabl’s original solution for tachyon condensation [2]:

$$\Psi_{\text{Schnabl}} = \lim_{N \rightarrow \infty} \left[\sum_{n=0}^N e^{K/2} c K B e^{nK} c e^{K/2} - e^{K/2} c B e^{NK} c e^{K/2} \right], \quad (4.12)$$

where the second term is the so-called phantom term. Applying our general formulae, we notice that the first term does not contribute to the parameters x and y because of the factor K in $c K B e^{nK} c$. This is consistent with Okawa’s observation [18] that the first term is a pure-gauge solution by itself. The phantom term makes a non-trivial contribution: $(x, y) = (-1, 0)$, and therefore, $|B_*(\Psi_{\text{Schnabl}})\rangle = 0$.

– **Simple solution for tachyon condensation** We then consider the simple solution for tachyon condensation [17]:

$$\Psi_{\text{ES}} = \frac{1}{\sqrt{1-K}} (c K B c - c) \frac{1}{\sqrt{1-K}}. \quad (4.13)$$

As discussed in [17], we can rewrite it as

$$\Psi = \lim_{\lambda \rightarrow 1^-} \Psi_\lambda - \frac{1}{\sqrt{1-K}} c B \tilde{\Omega}^\infty c \frac{1}{\sqrt{1-K}}, \quad (4.14)$$

where the second term is the phantom term. The state $\tilde{\Omega}^\infty$ is defined by $\tilde{\Omega}^\infty = \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\epsilon - K}$, and Ψ_λ is the following one parameter family of pure-gauge solutions:

$$\begin{aligned} \Psi_\lambda &= \left(1 - \lambda \frac{1}{\sqrt{1-K}} c B \frac{1}{\sqrt{1-K}} \right)^{-1} Q \left(1 - \lambda \frac{1}{\sqrt{1-K}} c B \frac{1}{\sqrt{1-K}} \right) \\ &= \lambda \frac{1}{\sqrt{1-K}} (c K B c - c) \frac{1}{\sqrt{1-K}} + \lambda \frac{1}{\sqrt{1-K}} c (1-K) \frac{1-\lambda}{1-\lambda-K} B c \frac{1}{\sqrt{1-K}}, \end{aligned} \quad (4.15)$$

which is well-defined for $0 \leq \lambda < 1$. Applying our general formulae, we again notice that the pure-gauge part $\lim_{\lambda \rightarrow 1^-} \Psi_\lambda$ does not contribute to the parameters x and y , and only the phantom term contributes: $(x, y) = (-1, 0)$, and therefore, $|B_*(\Psi_{\text{ES}})\rangle = 0$.

– **More general discussion** Finally, let us consider more general situation. In [53], Erler and Maccaferri gave an interpretation to the phantom term. For any non-trivial solution Ψ , we can define a non-trivial singular left gauge transformation U connecting to the perturbative vacuum [52]:

$$U\Psi = QU, \quad (4.16)$$

where U has a non-trivial kernel¹⁴ and its inverse is ill defined. Suppose that we can take a small number ϵ so that $\epsilon + U$ does not have any non-trivial kernel.¹⁵ Then, the inverse of $\epsilon + U$ is well defined, and we can rewrite Ψ using $(\epsilon + U)^{-1}$ as follows [52]:

$$\Psi = (\epsilon + U)^{-1}Q(\epsilon + U) + \frac{\epsilon}{\epsilon + U}\Psi. \quad (4.17)$$

Taking the limit $\epsilon \rightarrow 0$, we obtain

$$\Psi = \lim_{\epsilon \rightarrow 0} (\epsilon + U)^{-1}Q(\epsilon + U) + X^\infty \Psi, \quad (4.18)$$

where $X^\infty = \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\epsilon + U}$ is the projector onto the kernel of U called the boundary condition changing projector.¹⁶ Erler and Maccaferri gave an interpretation that the second term in (4.18) can be understood as the phantom term for known solutions and they showed its usefulness in the calculation of the energy and the gauge-invariant observables.

As we discussed, the phantom term of the solution for tachyon condensation [2, 17] determines the property of the boundary states. More generally in the KBc subalgebra, it is obvious from the expression (3.36) and (3.37) that the phantom term defined by $X^\infty \Psi$ determines the form of the boundary states: The parameters x and y are linear combinations of the contribution from each term of the solution. The first term in (4.18) is a pure-gauge solution by itself and it does not contribute to x and y . Then, only the phantom term contributes to the boundary states. Let us discuss the case of Okawa's formal solutions for example:

$$\Psi = F(K)c\frac{KB}{1 - F(K)^2}cF(K), \quad (4.19)$$

where we assume that $F(0)^2 = 1$ so that Ψ is not a pure-gauge solution. Since (4.19) can be written as

$$U\Psi = QU \quad \text{with} \quad U = 1 - F(K)cBF(K), \quad (4.20)$$

we obtain

$$\Psi = F(K)c\frac{KB}{1 + \epsilon - F(K)^2}cF(K) + F(K)cB\frac{K}{1 - F(K)^2}\frac{\epsilon}{\epsilon + 1 - F(K)^2}cF(K), \quad (4.21)$$

¹⁴ We regard a multiplication by the string field $U_{\tilde{\Psi}\Psi}$ as a morphism from a set of string fields to themselves [52].

¹⁵ For example, we take ϵ to be a negative real number when $U = K$.

¹⁶ See appendix A for more discussion on the projector and associated property in the KBc subalgebra.

where the first term is a pure-gauge solution by itself and the second term in the limit $\epsilon \rightarrow 0$ is the phantom term. Applying our general formulae, we notice that the first term does not contribute to x and y and only the second term contributes. It is also obvious that the contribution from the second term coincides with that from the solution (4.19) itself because $\lim_{z \rightarrow 0} \frac{z}{1 - F(z)^2} \frac{\epsilon}{\epsilon + 1 - F(z)^2} = \lim_{z \rightarrow 0} \frac{z}{1 - F(z)^2}$.

Summarizing above discussion, the boundary states for solutions in the KBc subalgebra are determined only from the phantom term $X^\infty \Psi$. Since our discussion highly depends on the expression of the formulae (3.36), (3.37), and (3.41) in the KBc subalgebra, it is not clear what is the situation for more general solutions such as marginal solutions [4, 27, 14]. It would be interesting to discuss the role of the phantom term in the calculation of the boundary states for them.

4.3.2 Proposed multiple D-brane solutions and their regularization

We then consider the multiple D-brane solutions proposed in [36, 39], which are in the class of Okawa's formal solutions:¹⁷

$$\Psi = F(K)c \frac{KB}{1 - F(K)^2} c F(K). \quad (4.22)$$

Although the formal solution (4.22) does not satisfy the regularity conditions in general, formally applying the general formulae, we obtain

$$x = \lim_{z \rightarrow 0} \left[\frac{1}{2} \frac{zF(z)^2}{1 - F(z)^2} - \frac{1}{2} \frac{z(1 - F(z)^2)'}{1 - F(z)^2} \right], \quad (4.23)$$

$$y = \lim_{z \rightarrow 0} \left[\frac{1}{2} \frac{zF(z)^2}{1 - F(z)^2} + \frac{1}{2} \frac{z(1 - F(z)^2)'}{1 - F(z)^2} \right]. \quad (4.24)$$

First, let us consider the state $|B_*(\Psi)\rangle$ in the limit $s \rightarrow 0$:

$$|B_*(\Psi)\rangle = (1 + x - y)|B\rangle + \mathcal{O}(s) = \left(1 - \lim_{z \rightarrow 0} \frac{z(1 - F(z)^2)'}{1 - F(z)^2} \right) |B\rangle + \mathcal{O}(s). \quad (4.25)$$

As discussed in [46], the state in the limit $s \rightarrow 0$ reproduces the boundary state for n D-branes when the function $F(z)$ satisfies the following property:

$$\lim_{z \rightarrow 0} z \frac{(1 - F(z)^2)'}{1 - F(z)^2} = 1 - n \quad \text{or} \quad 1 - F(z)^2 = a z^{1-n} + \dots, \quad (4.26)$$

¹⁷ It is straightforward to extend the following discussion to more general formal solutions (4.3). However, we concentrate on the Okawa's formal solutions for simplicity.

where a is a non-zero constant and the dots stand for higher order terms in z . This result in the limit $s \rightarrow 0$ is essentially equivalent to the calculation of the gauge-invariant observables in [36, 39]. However, the situation is different for finite s . When $F(z)$ satisfies (4.26), the parameters x and y are given by

$$(x, y) = \begin{cases} \left(\frac{n-1}{2}, -\frac{n-1}{2} \right) & \text{for } n > 0, \\ \left(a - \frac{1}{2}, a + \frac{1}{2} \right) & \text{for } n = 0, \\ (\text{singular}, \text{singular}) & \text{for } n < 0. \end{cases} \quad (4.27)$$

Since the closed string state $|B_*(\Psi)\rangle$ has a non-trivial s -dependence unless $x = y = 0$ or $x = y - 1$, that for (4.27) does not reproduce an appropriate boundary state when $n \neq 0, 1$.

In the following, we try to improve this formal observation using two types of concrete regularization.¹⁸ We mainly consider the regularization for the following double brane solution for simplicity:

$$\Psi_{\text{double}} = -\frac{1}{\sqrt{-K}} c \frac{K^2}{K-1} B c \frac{1}{\sqrt{-K}}, \quad (4.28)$$

where we chose $F(K) = 1/\sqrt{-K}$.

– **K_ϵ regularization** We first consider the so-called K_ϵ regularization discussed in [39, 44]:

$$\Psi_{\text{HKMS}} = -\lim_{\epsilon \rightarrow 0} \frac{1}{\sqrt{\epsilon - K}} c \frac{(K - \epsilon)^2}{K - \epsilon - 1} B c \frac{1}{\sqrt{\epsilon - K}}, \quad (4.29)$$

where all K 's in (4.28) are replaced by $K - \epsilon$. In [39, 44], it was shown that (4.29) reproduces the energy and the gauge-invariant observables for double branes. It also satisfies the equation of motion when contracted to the solution itself. However, the equation of motion is not satisfied when contracted to some states in the Fock space [39].

Then, let us calculate the boundary states using K_ϵ regularization. We consider the K_ϵ regularization in the following general setting:

$$\Psi = \lim_{\epsilon \rightarrow 0} \Psi_\epsilon \quad \text{with} \quad \Psi_\epsilon = F(K - \epsilon) c \frac{(K - \epsilon)B}{1 - F(K - \epsilon)^2} c F(K - \epsilon). \quad (4.30)$$

Applying our general formulae, we obtain

$$x_\epsilon = \frac{1}{2} \frac{(-\epsilon) F(-\epsilon)^2}{1 - F(-\epsilon)^2} - \frac{1}{2} \frac{(-\epsilon) (1 - F(-\epsilon)^2)'}{1 - F(-\epsilon)^2}, \quad (4.31)$$

¹⁸ As we mentioned in section 3.3, it is not clear whether the Schnabl gauge calculation is justified in the singular limit. However, in this subsection, we formally apply our results based on the Schnabl gauge calculation in order to discuss the relation to the regularizations in [39, 44, 45] of proposed multiple D-brane solutions.

$$y_\epsilon = \frac{1}{2} \frac{(-\epsilon) F(-\epsilon)^2}{1 - F(-\epsilon)^2} + \frac{1}{2} \frac{(-\epsilon) (1 - F(-\epsilon)^2)'}{1 - F(-\epsilon)^2}. \quad (4.32)$$

Here, these expressions coincide with those in the formal discussion (4.23) and (4.24). Therefore, the calculation of boundary states in the K_ϵ regularization results in the formal discussion (4.27). For the double brane solution (4.29), the state $|B_*(\Psi)\rangle$ is given by

$$|B_*(\Psi)\rangle = \lim_{\epsilon \rightarrow 0} |B_*(\Psi_\epsilon)\rangle = (e^{\frac{s}{2}} + e^{-\frac{s}{2}})|B\rangle = 2|B\rangle + \mathcal{O}(s). \quad (4.33)$$

This result is consistent with the results in [39]: The non-trivial s -dependence implies the violation of the equation of motion. The calculation of the state $|B_*(\Psi)\rangle$ in the $s \rightarrow 0$ limit, which is essentially equivalent to that of the gauge-invariant observables, reproduces the boundary states for double branes.

It is also notable that Ψ_ϵ can be written as follows:

$$\Psi_\epsilon = F(K - \epsilon)c \frac{KB}{1 - F(K - \epsilon)^2} cF(K - \epsilon) - F(K - \epsilon)c \frac{\epsilon B}{1 - F(K - \epsilon)^2} cF(K - \epsilon), \quad (4.34)$$

where the first term is a pure-gauge solution by itself and the second term looks like a phantom term. Although the second term leads to the violation of the equation of motion for double brane solutions, it reproduces the correct phantom term for solutions for tachyon condensation. For example, the second term for the simple solution for tachyon condensation is given by

$$(\text{second term}) = -\frac{1}{\sqrt{1 + \epsilon - K}} c(1 + K - \epsilon) \frac{\epsilon}{\epsilon - K} Bc \frac{1}{\sqrt{1 + \epsilon - K}}. \quad (4.35)$$

In the limit $\epsilon \rightarrow 0$, it reproduces the correct phantom term:¹⁹

$$\begin{aligned} (\text{second term}) &\rightarrow -\lim_{\epsilon \rightarrow 0} \left[\frac{1}{\sqrt{1 + \epsilon - K}} c(1 + \epsilon - K) \frac{\epsilon}{\epsilon - K} Bc \frac{1}{\sqrt{1 + \epsilon - K}} \right] \\ &= -\frac{1}{\sqrt{1 - K}} c \tilde{\Omega}^\infty Bc \frac{1}{\sqrt{1 - K}}. \end{aligned} \quad (4.36)$$

Then, the same discussion as that in section 4.3.1 holds.

– **Regularization in [45]** We then consider another regularization discussed in [45] by one of the authors for the double brane solution (4.28):

$$\Psi_{\text{TM}} = \Psi_{R_0} - \varphi_p, \quad (4.37)$$

¹⁹ We ignore the $1 + \epsilon - K$ factor since this would give a subleading contribution to the phantom piece as discussed in [17].

where

$$\Psi_{R_0} = - \lim_{\Lambda \rightarrow \infty} \int_0^\infty R_0(\Lambda; x) e^{Kx} dx c \frac{K^2}{K-1} Bc \quad \text{with} \quad R_0(\Lambda; x) = 1 - \frac{\ln(x+1)}{\ln(\Lambda+1)}, \quad (4.38)$$

and φ_p is a correction term whose expression is presented in [45]. In [45], it was shown that the equation of motion for (4.37) is satisfied when contracted to the state in the Fock space and the solution itself, and the solution (4.37) reproduces the expected energy for double branes. However, it was also shown that the gauge-invariant observables [43] for (4.37) are not those for the double brane background but those for the perturbative vacuum.

Then, let us calculate the boundary states using this regularization. Applying our general formulae, we obtain

$$x = y = 0, \quad (4.39)$$

and therefore,

$$|B_*\rangle = |B\rangle, \quad (4.40)$$

which is the boundary state for the perturbative vacuum. Although the solution (4.37) does not reproduce the boundary state for double branes, our result is not inconsistent with the results in [45]: The boundary state is s -independent while the solution satisfies the equation of motion. Both of the boundary state and the gauge-invariant observables reproduce those for the perturbative vacuum. However, it is mysterious that the solution (4.37) reproduces the boundary state and the gauge-invariant observables for the perturbative vacuum while it reproduces the energy for double branes.

5 Another class of formal solutions

In the previous section we showed that the formal solutions based on the pure-gauge ansatz (4.4) do not reproduce the boundary state for the ghost brane solution. In this section we discuss another class of formal solutions.

Suppose that Ψ is a linear combination of two Okawa's formal solutions Ψ_1 and Ψ_2 :

$$\Psi = \Psi_1 + \Psi_2 \quad (5.1)$$

with

$$\Psi_1 = F(K) c \frac{KB}{1 - F(K)^2} c F(K), \quad \Psi_2 = H(K) c \frac{KB}{1 - H(K)^2} c H(K). \quad (5.2)$$

Formally using the equations of motion for Ψ_1 and Ψ_2 , the equation of motion for Ψ is reduced to

$$Q\Psi + \Psi^2 = Q\Psi_1 + \Psi_1^2 + Q\Psi_2 + \Psi_2^2 + \Psi_1\Psi_2 + \Psi_2\Psi_1 = \Psi_1\Psi_2 + \Psi_2\Psi_1, \quad (5.3)$$

and the cross terms in (5.3) vanish when $F(K)H(K) = 1$ because $c^2 = 0$.²⁰ We then obtain the following class of formal solutions:

$$\Psi = F(K)c\frac{KB}{1-F(K)^2}cF(K) + H(K)c\frac{KB}{1-H(K)^2}cH(K) \quad (5.5)$$

with

$$F(K)H(K) = 1, \quad (5.6)$$

where we assume that $F(K)$ and $H(K)$ satisfy the regularity conditions I and II introduced in section 2.2, and $K/(1-F^2)$ and $K/(1-H^2)$ satisfy the regularity condition I. Here we note that this class of formal solutions cannot be written in the pure-gauge form (4.4) as discussed in appendix B.

We also note that the formal solution (5.5) seems to contain some identity-based terms: For simplicity, let $F(K)^2$ be a meromorphic function of K . Then, either F^2 or $1/F^2$ inevitably becomes an improper fraction, which contains identity-based terms. This in turn makes the resulting solution (5.5) contain some identity-based terms. Therefore, we need careful treatment of these formal solutions.

In the following, we classify this class of solutions with respect to boundary states and we propose a concrete example of the ghost brane solution.

5.1 Boundary states

We start from the calculation of the closed string state $|B_*(\Psi)\rangle$ for the formal solutions (5.5). Using the general formula, we obtain

$$\begin{aligned} x &= \lim_{z \rightarrow 0} \left[\frac{z}{1-F(z)^2} \left(\frac{1}{2}F(z)^2 + F(z)F'(z) \right) + \frac{z}{1-H(z)^2} \left(\frac{1}{2}H(z)^2 + H(z)H'(z) \right) \right] \\ &= -\frac{1}{2} \lim_{z \rightarrow 0} \left[\frac{z(1-F(z)^2)'}{1-F(z)^2} + \frac{z(1-H(z)^2)'}{1-H(z)^2} \right]. \end{aligned} \quad (5.7)$$

In the same way, y is evaluated as

$$y = \frac{1}{2} \lim_{z \rightarrow 0} \left[\frac{z(1-F(z)^2)'}{1-F(z)^2} + \frac{z(1-H(z)^2)'}{1-H(z)^2} \right]. \quad (5.8)$$

²⁰ Here we can alternatively take $F(K)H(K) = a$, where a is an arbitrary nonzero constant. However, the resulting solution Ψ_a for $a \neq 1$,

$$\Psi_a = F(K)c\frac{KB}{1-F(K)^2}cF(K) + (aF(K)^{-1})c\frac{KB}{1-(aF(K)^{-1})^2}c(aF(K)^{-1}), \quad (5.4)$$

can be written in the form (4.4).

We notice that the function $1 - F(z)^2$ determines the property of the formal solution. Let us consider the following two cases.

- $F(0) \neq 1$, and therefore, $H(z) \neq 1$

The regularity conditions imply $x = y = 0$, and therefore, $|B_*(\Psi)\rangle = |B\rangle$. The boundary state for the perturbative vacuum can be reproduced by this class of formal solutions. However, they cannot be written in the pure-gauge form (4.4), and they seem not to be gauge equivalent to the perturbative vacuum solution.

- $F(0) = 1$, and therefore, $H(0) = 1$

The regularity conditions imply $1 - F(z)^2 = az + \mathcal{O}(z^2)$ and $1 - H(z)^2 = -az + \mathcal{O}(z^2)$, where a is a non-zero constant. We then obtain $x = -1$ and $y = 1$. Therefore, $|B_*(\Psi)\rangle = -|B\rangle$, which is nothing but the boundary state for the ghost D-brane.

Therefore, we conclude that the following boundary states can be reproduced by the formal solutions of the form (5.5): $|B_*(\Psi)\rangle = |B\rangle$ and $-|B\rangle$.

5.2 A candidate for the ghost brane solution

In the last subsection we showed that the formal solution (5.5) reproduces the boundary state $-|B\rangle$ when $1 - F(z)^2 = az + \mathcal{O}(z^2)$. In this subsection we consider a particular choice of the function $F(z)$. For simplicity, let $F(K)^2$ be a meromorphic function of K , and we set

$$F(K) = \sqrt{\frac{1-pK}{1-qK}} = \int_0^\infty dt_1 \int_0^\infty dt_2 \frac{e^{-t_1}}{\sqrt{\pi t_1}} \frac{e^{-t_2}}{\sqrt{\pi t_2}} e^{(qt_1+pt_2)K} (1-pK) \quad (5.9)$$

to obtain²¹

$$\Psi_{ghost} = \sqrt{\frac{1-pK}{1-qK}} c \frac{1-qK}{p-q} Bc \sqrt{\frac{1-pK}{1-qK}} + \sqrt{\frac{1-qK}{1-pK}} c \frac{1-pK}{q-p} Bc \sqrt{\frac{1-qK}{1-pK}}, \quad (5.10)$$

where p and q are distinct positive constants. As we mentioned, the formal solutions (5.10) contain some identity based terms. They become more explicit when we introduce a non-real solution $\tilde{\Psi}_{ghost}$

$$\begin{aligned} \tilde{\Psi}_{ghost} &= \sqrt{\frac{1-pK}{1-qK}} \Psi_{ghost} \sqrt{\frac{1-qK}{1-pK}} \\ &= \frac{1-pK}{1-qK} c \frac{1-qK}{p-q} Bc + c \frac{1-pK}{q-p} Bc \frac{1-qK}{1-pK}, \end{aligned} \quad (5.11)$$

²¹ Each term in (5.10) takes a similar form as the solution for the tachyon vacuum discussed in [54, 55].

which can be written in the following form:

$$\tilde{\Psi}_{ghost} = \frac{1}{q} \frac{1}{1-qK} c(qK-1)Bc + \frac{1}{p} c(pK-1)Bc \frac{1}{1-pK} + \frac{p+q}{pq} c - cKBc. \quad (5.12)$$

Here the first two terms are some variants of the simple solutions for tachyon condensation, and the last two terms are identity-based terms. (It seems that, for any choice of $F(K)$, ghost brane solutions of the form (5.5) always contain identity-based term as long as we assume our regularity conditions.) Since the expression (5.12) contains identity-based terms, we need careful treatment when we discuss the equation of motion and the energy density of the solution.

– **Energy** Let us calculate the energy of the solution (5.10). We use the expression (5.11)²² and we define

$$\tilde{\Psi}_1 = \frac{1-pK}{1-qK} c \frac{1-qK}{p-q} Bc, \quad \tilde{\Psi}_2 = c \frac{1-pK}{q-p} Bc \frac{1-qK}{1-pK}. \quad (5.13)$$

Then, we can express the energy as a superposition of the correlation functions of wedge-based states as follows:

$$\begin{aligned} & \langle \tilde{\Psi}_1 Q \tilde{\Psi}_1 \rangle \\ &= \frac{1}{q^2(p-q)^2} \int_0^\infty dx \int_0^\infty dy e^{-(x+y)/q} \\ & \quad \times (1-p\partial_x)(1-p\partial_y)(1-q\partial_u)(1-q\partial_v) \langle e^{Kx} c e^{Ku} Bc Q(e^{Ky} c e^{Kv} Bc) \rangle \Big|_{u \rightarrow 0, v \rightarrow 0} \end{aligned} \quad (5.14)$$

$$\begin{aligned} &= \frac{1}{q^2(p-q)^2} \int_0^\infty dx \int_0^\infty dy e^{-(x+y)/q} \\ & \quad \times \left[\frac{(4\pi^2 p^2 xy + (x+y)^2 (2(p-x-y)^2 - (x+y)^2)) \cos\left(\frac{2\pi x}{x+y}\right)}{2\pi^2 (x+y)^2} \right. \\ & \quad \left. - \frac{2(p-x-y)^2 - (x+y)^2}{2\pi^2} - \frac{p(x-y)(-p+x+y) \sin\left(\frac{2\pi x}{x+y}\right)}{\pi(x+y)} \right] \end{aligned} \quad (5.15)$$

$$= -\frac{3}{\pi^2}, \quad (5.16)$$

where the expression of the integration kernel, $\langle e^{Kx} c e^{Ku} Bc Q(e^{Ky} c e^{Kv} Bc) \rangle$, is presented in [45]. Note that there exist no terms proportional to $\delta(x)\delta(y)$ in the integrand of (5.15), which are contribution from the identity-based terms. Similarly, we obtain

$$\langle \tilde{\Psi}_2 Q \tilde{\Psi}_2 \rangle = -\frac{3}{\pi^2}, \quad (5.17)$$

²² We note that Ψ and $\tilde{\Psi}$ have the same energy density because $\langle \Psi Q \Psi \rangle = \langle \tilde{\Psi} Q \tilde{\Psi} \rangle$ and $\langle \Psi^3 \rangle = \langle \tilde{\Psi}^3 \rangle$.

$$\langle \tilde{\Psi}_1 Q \tilde{\Psi}_2 \rangle = 0. \quad (5.18)$$

Since correlation functions on a cylinder of zero circumference have some ambiguity in general, we also confirm the following limit:

$$\lim_{\epsilon_1=a\epsilon_2 \rightarrow 0} \langle \tilde{\Psi}_1 e^{\epsilon_1 K} Q \tilde{\Psi}_2 e^{\epsilon_2 K} \rangle = 0 \quad \text{for } \forall a > 0, \quad (5.19)$$

which reproduces the result in (5.18). Therefore, we conclude that we can calculate the energy of the solution (5.10) without ambiguity unlike the identity based solutions²³, and the resulting energy density is

$$E = -\frac{1}{\pi^2}, \quad (5.21)$$

which coincides with the energy density of the ghost D-brane [49].

– **Properties** We summarize our current understanding of our ghost brane solution below:

- Its component fields are all finite: $\langle \varphi, \Psi_{ghost} \rangle = \text{finite}$.
- It satisfies the equation of motion when contracted to the state in the Fock space and the solution itself: $\langle \varphi, Q\Psi_{ghost} + \Psi_{ghost}^2 \rangle = 0$ and $\langle \Psi_{ghost}, Q\Psi_{ghost} + \Psi_{ghost}^2 \rangle = 0$.
- Its energy density is twice as that of the tachyon vacuum solution: $E = -\frac{1}{\pi^2}$, which coincides with that of the ghost D-brane [49].
- It reproduces the boundary state for the ghost D-brane: $|B_*(\Psi_{ghost})\rangle = -|B\rangle$.
- It satisfies the so-called weak consistency condition introduced in [52].²⁴

At this point, it is not clear whether the solution can be regarded as a physical solution or not. All we can say is that the properties itemized above seem not to prohibit it. Further investigation would be required.

²³We should note that other regular solutions such as the simple solution for tachyon condensation [17] can be written in the form containing identity-based terms: it can be written as

$$\Psi_{ES} = \frac{1}{1-K} c(K-1)Bc = \frac{1}{1-K} Bc(1-K)c - c. \quad (5.20)$$

²⁴ See appendix A for details.

6 Discussion

In this paper we evaluated boundary states for general string fields in the KBc subalgebra under some regularity conditions. By requiring the s -independence as a necessary condition to satisfy the equation of motion, we showed that there are only three possible boundary states in our class: $|B_*\rangle = \pm|B\rangle$ and 0. Our results seem to suggest that we are not able to construct multiple D-brane solutions in the KBc subalgebra without relaxing our regularity conditions. However, as we discussed in section 3.3, even if we consider singular string fields such as proposed multiple D-brane solutions, the expected boundary states cannot be reproduced as long as we express them as a limit of regular string fields and formally use our formula (3.36), (3.37), and (3.41).

To be precise, our derivation was based on the Schnabl gauge calculation. As discussed in [50], Schnabl gauge can be understood as a singular limit of a one parameter family of regular linear b gauges called λ -regularized gauges. Originally in [47], the Schnabl gauge calculation was introduced based on the discussion [51] using λ -regularized gauges. Although the calculation seems to be valid for regular string fields, it is not clear whether it is applicable for singular string fields. It requires careful investigations using λ -regularized gauges in order to clarify this point and complete our discussion. Some preliminary works in this direction are underway and we hope to report our progress elsewhere.

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Note added

The contents of appendix B.2 are added in the replacement from v1 to v2 on arXiv. While we were preparing the replacement, the paper [56] appeared on arXiv, in which a level expansion method in the sliver frame was proposed and the homomorphism of the KBc subalgebra in appendix B.2 was also pointed out in that context. We would like to thank Ted Erler for correspondence.

A Projectors and consistency conditions

In [52], extending Ellwood's discussion [32] on singular gauge transformations in open string field theory, Erler and Maccaferri showed that there is always a nonzero left gauge transformation $U_{\tilde{\Psi}\Psi}$ connecting any pair of classical solutions Ψ and $\tilde{\Psi}$:

$$QU_{\tilde{\Psi}\Psi} + \tilde{\Psi}U_{\tilde{\Psi}\Psi} = U_{\tilde{\Psi}\Psi}\Psi. \quad (\text{A.1})$$

When Ψ and $\tilde{\Psi}$ describe different backgrounds, $U_{\tilde{\Psi}\Psi}$ has a non-trivial kernel, and it is expected that the kernel of $U_{\tilde{\Psi}\Psi}$ captures some properties of the solutions. They introduced a projector $X_{\tilde{\Psi}\Psi}^\infty$ onto the kernel and carefully investigated its property. They found that the projectors for known solutions have interesting structures associated with the BCFT described by the solutions and they called the projector $X_{\tilde{\Psi}\Psi}^\infty$ the boundary condition changing projector. Although its property is not fully understood yet, we expect that it would be helpful for constructing new solutions.²⁵ In this appendix we first investigate the structure of the projector in the KBc subalgebra, and then, we apply the result to the ghost brane solution (5.10). We also show that the ghost brane solution satisfy the so-called weak consistency condition introduced in [52].

A.1 Projectors in KBc subalgebra

As discussed in [52], for any solutions Ψ and $\tilde{\Psi}$, we can find a non-zero left gauge transformation $U_{\tilde{\Psi}\Psi}$ in the following way:

$$U_{\tilde{\Psi}\Psi} = Q\mathbf{b} + \tilde{\Psi}\mathbf{b} + \mathbf{b}\Psi, \quad (\text{A.2})$$

where \mathbf{b} is a Grassmann-odd state of ghost number -1 . The relation (A.1) follows from the equations of motion for Ψ and $\tilde{\Psi}$. In this paper we define the projector $X_{\tilde{\Psi}\Psi}^\infty$ onto the kernel of $U_{\tilde{\Psi}\Psi}$ by²⁶

$$X_{\tilde{\Psi}\Psi}^\infty = \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\epsilon + U_{\tilde{\Psi}\Psi}}. \quad (\text{A.3})$$

In this subsection we calculate $U_{\tilde{\Psi}\Psi}$ defined by (A.2) and the associated projector $X_{\tilde{\Psi}\Psi}^\infty$ for general string fields in the KBc subalgebra²⁷:

$$\Psi = \sum_i F_i(K)cBG_i(K)cH_i(K), \quad \tilde{\Psi} = \sum_i \tilde{F}_i(K)cB\tilde{G}_i(K)c\tilde{H}_i(K). \quad (\text{A.4})$$

We note that $U_{\tilde{\Psi}\Psi}$ and $X_{\tilde{\Psi}\Psi}^\infty$ are well-defined by (A.2) and (A.3) although Ψ and $\tilde{\Psi}$ in (A.4) do not necessarily satisfy the equation of motion and $U_{\tilde{\Psi}\Psi}$ do not necessarily reproduce the left gauge transformation (A.1).

²⁵ See [53] for an interesting application of the projector to determining the so-called phantom term.

²⁶ We assume that we can take a small number ϵ so that $\epsilon + U$ does not have any non-trivial kernel. For example, we take ϵ to be a negative real number when $U = K$.

²⁷ In the calculation of this appendix, we assume that $F_i(K)$, $G_i(K)$, $H_i(K)$, $\tilde{F}_i(K)$, $\tilde{G}_i(K)$, and $\tilde{H}_i(K)$ satisfy the regularity condition I introduced in (2.10).

A.1.1 General form of the projector

In the KBc subalgebra, the general form of the state \mathbf{b} of ghost number -1 is given by

$$\mathbf{b} = BM(K), \quad (\text{A.5})$$

where $M(K)$ is a function of K satisfying the regularity condition I and we also assume that it does not have a non-trivial kernel. Substituting this general form into the definition (A.2), we obtain

$$\begin{aligned} U_{\tilde{\Psi}\Psi} &= KM + \sum_i \tilde{F}_i cB \tilde{G}_i \tilde{H}_i M + M \sum_i F_i G_i BcH_i \\ &= (K + \sum_i \tilde{F}_i \tilde{G}_i \tilde{H}_i)M - \sum_i \tilde{F}_i Bc \tilde{G}_i \tilde{H}_i M + M \sum_i F_i G_i BcH_i. \end{aligned} \quad (\text{A.6})$$

It is convenient to introduce the following formula in the KBc subalgebra:

$$(H + \sum_i I_i BcJ_i)^{-1} = \frac{1}{H + \sum_i I_i J_i} (1 + \sum_j I_j cB J_j H^{-1}), \quad (\text{A.7})$$

where H , I_i , and J_i are functions of K satisfying $H \neq 0$ and $H + \sum_i I_i J_i \neq 0$. Using this formula, we calculate $(\epsilon + U_{\tilde{\Psi}\Psi})^{-1}$ as

$$\begin{aligned} (\epsilon + U_{\tilde{\Psi}\Psi})^{-1} &= \frac{1}{\epsilon + (K + \sum_i F_i G_i H_i)M} \\ &\quad - \frac{1}{\epsilon + (K + \sum_i F_i G_i H_i)M} \left(\sum_j \tilde{F}_j cB \tilde{G}_j \tilde{H}_j M \right) \frac{1}{\epsilon + (K + \sum_k \tilde{F}_k \tilde{G}_k \tilde{H}_k)M} \\ &\quad + \frac{1}{\epsilon + (K + \sum_i F_i G_i H_i)M} \left(M \sum_j F_j G_j cB H_j \right) \frac{1}{\epsilon + (K + \sum_k \tilde{F}_k \tilde{G}_k \tilde{H}_k)M}. \end{aligned} \quad (\text{A.8})$$

Then, the associated projector $X_{\tilde{\Psi}\Psi}^\infty$ is given by

$$X_{\tilde{\Psi}\Psi}^\infty = \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\epsilon + U_{\tilde{\Psi}\Psi}} = \mathcal{P}_\Psi + \mathcal{P}_{\Psi\tilde{\Psi}}^{cB}, \quad (\text{A.9})$$

where

$$\mathcal{P}_\Psi = \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\epsilon + (K + \sum_i F_i G_i H_i)M}, \quad (\text{A.10})$$

$$\begin{aligned} \mathcal{P}_{\Psi\tilde{\Psi}}^{cB} &= \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\epsilon + (K + \sum_i F_i G_i H_i)M} \left(M \sum_j F_j G_j cB H_j \right) \frac{1}{\epsilon + (K + \sum_k \tilde{F}_k \tilde{G}_k \tilde{H}_k)M} \\ &\quad - \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\epsilon + (K + \sum_i F_i G_i H_i)M} \left(\sum_j \tilde{F}_j cB \tilde{G}_j \tilde{H}_j M \right) \frac{1}{\epsilon + (K + \sum_k \tilde{F}_k \tilde{G}_k \tilde{H}_k)M}. \end{aligned} \quad (\text{A.11})$$

We notice that the property of the projector is determined by $\sum_i F_i G_i H_i$ and $\sum_i \tilde{F}_i \tilde{G}_i \tilde{H}_i$.

A.1.2 From tachyon vacuum to general string fields

Let us consider the case when $\tilde{\Psi}$ is a solution for the tachyon vacuum:

$$\tilde{\Psi} = \Psi_{\text{tv}} = I(K)c\frac{KB}{1-I(K)^2}cI(K), \quad (\text{A.12})$$

where $I(K) = e^{K/2}$ for Schnabl's solution [2] and $I(K) = \frac{1}{\sqrt{1-K}}$ for the simple solution [17].

In this case, $\mathcal{P}_{\Psi\tilde{\Psi}}^{cB}$ is given by

$$\mathcal{P}_{\Psi\tilde{\Psi}}^{cB} = \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\epsilon + M(K + \sum_i F_i G_i H_i)} \left(M \sum_i F_i G_i c B H_i - I c B \frac{K}{1-I^2} I M \right) \frac{1}{\frac{K}{1-I^2} M + \epsilon}. \quad (\text{A.13})$$

We notice that we can take the limit $\lim_{\epsilon \rightarrow 0} (\frac{K}{1-I^2} M + \epsilon)^{-1} = \frac{1-I^2}{K} M^{-1}$ without using singular expression because $\frac{K}{1-I^2} M$ does not have a non-trivial kernel. Therefore, we can rewrite the projector $X_{\Psi_{\text{tv}}\Psi}^\infty$ as

$$X_{\Psi_{\text{tv}}\Psi}^\infty = \mathcal{P}_\Psi + \mathcal{P}_\Psi M \sum_i F_i G_i c B H_i \frac{1-I^2}{KM} - \mathcal{P}_\Psi I c B I. \quad (\text{A.14})$$

A.1.3 From perturbative vacuum to general string fields

In the case when $\tilde{\Psi}$ is a solution for the perturbative vacuum

$$\tilde{\Psi} = \Psi_{\text{pv}} = 0, \quad (\text{A.15})$$

the projector $X_{\Psi_{\text{pv}}\Psi}^\infty$ is given by

$$X_{\Psi_{\text{pv}}\Psi}^\infty = \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\epsilon + M(K + \sum_i F_i G_i H_i)} + \lim_{\epsilon \rightarrow 0} \frac{M}{\epsilon + M(K + \sum_j F_j G_j H_j)} \sum_i F_i G_i c B H_i \frac{\epsilon}{\epsilon + KM}. \quad (\text{A.16})$$

Unlike the previous case, the limit $\epsilon \rightarrow 0$ can be singular in general because the state K has a non-trivial kernel.

A.2 Consistency condition for the ghost brane solution

In this subsection we apply our result in the previous subsection to the ghost brane solution (5.10), and we show that it satisfies the so-called weak consistency condition [52].

A.2.1 Consistency conditions

When two solutions Ψ and $\tilde{\Psi}$ are connected by a left gauge transformation $U_{\tilde{\Psi}\Psi}$, it follows from the expression (A.1) that

$$\text{Im} (Q + \tilde{\Psi})U_{\tilde{\Psi}\Psi} \subseteq \text{Im} U_{\tilde{\Psi}\Psi}, \quad (\text{A.17})$$

which is called the strong consistency condition [52]. Provided that the projector $X_{\tilde{\Psi}\Psi}^\infty$ onto the kernel of $U_{\tilde{\Psi}\Psi}$ exists, $X_{\tilde{\Psi}\Psi}^\infty$ satisfies

$$\ker U_{\tilde{\Psi}\Psi} = \text{Im} X_{\tilde{\Psi}\Psi}^\infty, \quad \text{Im} U_{\tilde{\Psi}\Psi} \subseteq \ker X_{\tilde{\Psi}\Psi}^\infty, \quad (\text{A.18})$$

and therefore, the following relation is implied:

$$\text{Im} (Q + \tilde{\Psi})U_{\tilde{\Psi}\Psi} \subseteq \text{Im} U_{\tilde{\Psi}\Psi} \subseteq \ker X_{\tilde{\Psi}\Psi}^\infty. \quad (\text{A.19})$$

In other words,

$$X_{\tilde{\Psi}\Psi}^\infty Q_{\tilde{\Psi}} U_{\tilde{\Psi}\Psi} = 0, \quad (\text{A.20})$$

which is called the weak consistency condition [52]. The relation (A.20) should hold if two solutions are connected by left gauge transformation and the projector $X_{\tilde{\Psi}\Psi}^\infty$ exists. In what follows, we show that the ghost brane solution (5.10) satisfies the weak consistency condition (A.20) when connected to the tachyon vacuum and the perturbative vacuum.

A.2.2 From tachyon vacuum to the ghost brane

Let us start from the case of the tachyon vacuum. Using (A.6) and (A.14), the gauge transformation $U_{\Psi_{\text{tv}}\Psi_{\text{gh}}}$ and the projector $X_{\Psi_{\text{tv}}\Psi_{\text{gh}}}^\infty$ are given by

$$U_{\Psi_{\text{tv}}\Psi_{\text{gh}}} = \frac{K}{1-I^2}M - IBc\frac{K}{1-I^2}IM + MF\frac{K}{1-F^2}BcF(1-H^2), \quad (\text{A.21})$$

$$X_{\Psi_{\text{tv}}\Psi_{\text{gh}}}^\infty = 1 + MF\frac{K}{1-F^2}cBF(1-H^2)\frac{1-I^2}{KM} - IcBI. \quad (\text{A.22})$$

We then obtain that

$$QU_{\Psi_{\text{tv}}\Psi_{\text{gh}}} + \Psi_{\text{tv}}U_{\Psi_{\text{tv}}\Psi_{\text{gh}}} = MF\frac{K}{1-F^2}cBKcF(1-H^2) + Ic\frac{K}{1-I^2}IMF\frac{K}{1-F^2}BcF(1-H^2). \quad (\text{A.23})$$

It is straightforward to show that

$$X_{\Psi_{\text{tv}}\Psi_{\text{gh}}}^\infty Q_{\Psi_{\text{tv}}} U_{\Psi_{\text{tv}}\Psi_{\text{gh}}} = 0, \quad (\text{A.24})$$

and therefore, the ghost brane solution satisfies the weak consistency condition when connected to the tachyon vacuum.

A.2.3 From perturbative vacuum to the ghost brane

We then consider the case of the perturbative vacuum. Using (A.6) and (A.14), the gauge transformation $U_{\Psi_{\text{tv}}\Psi_{\text{gh}}}$ and the projector $X_{\Psi_{\text{tv}}\Psi_{\text{gh}}}^\infty$ are given by²⁸

$$U_{\Psi_{\text{pv}}\Psi_{\text{gh}}} = KM + M \frac{K}{1-F^2} BcF(1-H^2), \quad (\text{A.25})$$

$$\begin{aligned} X_{\Psi_{\text{pv}}\Psi_{\text{gh}}}^\infty &= 1 + \lim_{\epsilon \rightarrow 0} MF \frac{K}{1-F^2} cBF(1-H^2) \frac{1}{\epsilon + KM} \\ &= 1 + MF \frac{K}{1-F^2} cBF \frac{1-H^2}{K} KM^{-1}(1-\tilde{\Omega}^\infty), \end{aligned} \quad (\text{A.26})$$

where we used

$$\lim_{\epsilon \rightarrow 0} (1-H^2) \frac{1}{\epsilon + KM} = \frac{1-H^2}{K} M^{-1} \lim_{\epsilon \rightarrow 0} \frac{KM}{\epsilon + KM} = \frac{1-H^2}{K} M^{-1} (1-\tilde{\Omega}^\infty). \quad (\text{A.27})$$

We then obtain that

$$QU_{\Psi_{\text{pv}}\Psi_{\text{gh}}} = MF \frac{K}{1-F^2} cBKcF(1-H^2). \quad (\text{A.28})$$

It is straightforward to show that

$$\begin{aligned} &X_{\Psi_{\text{pv}}\Psi_{\text{gh}}}^\infty QU_{\Psi_{\text{pv}}\Psi_{\text{gh}}} \\ &= MF \frac{K}{1-F^2} cBKcF(1-H^2) \\ &\quad + MF \frac{K}{1-F^2} cBF \frac{1-H^2}{K} M^{-1} (1-\tilde{\Omega}^\infty) MF \frac{K}{1-F^2} cBKcF(1-H^2) \\ &= MF \frac{K}{1-F^2} cBKcF(1-H^2) - MF \frac{K}{1-F^2} cBKcF(1-H^2) \\ &= 0, \end{aligned} \quad (\text{A.29})$$

where we used $FH = 1$ and formally used $\Omega^\infty K = 0$. Therefore, the ghost brane solution satisfies the weak consistency condition when connected to the perturbative vacuum.²⁹

B Algebraic structure

The results obtained in this paper can be regarded as classification of classical solutions in the KBc subalgebra with respect to corresponding boundary states. On the other hand, works by

²⁸ We use the symbol $\tilde{\Omega}^\infty$ to denote sliver-like states as in section 4.3.1.

²⁹ In [52], it was shown that another candidate for the ghost brane solution, which was constructed in a similar way as the proposed multiple D-brane solutions [36, 39], does not satisfy the weak consistency condition when connected to the tachyon vacuum. However, our solution is constructed in a different way, and it satisfies both the equation of motion and the weak consistency condition.

Erler [24] and Murata and Schnabl [36, 39] are classification with respect to the energy. In this short appendix, we attempt to clarify the underlying algebraic structure in these discussions from the viewpoint of gauge structures. We also provide a proof that the solution in the class (5.5) cannot be written in the pure-gauge form (4.4).

B.1 On the relation between (4.4) and (5.5)

Let Ψ be a solution to the equation of motion. Define the gauge parameter U by³⁰

$$U = 1 + A\Psi, \quad (\text{B.1})$$

where A denotes the formal homotopy operator around the perturbative vacuum, $A = B/K$. Supposing that the inverse of U exists, we can rewrite Ψ as follows:³¹

$$\Psi = U^{-1}QU. \quad (\text{B.2})$$

Note that the normalization of the gauge parameter U is different from that in section 4. Then consider when the inverse of U exist. The answer is already presented in section 4; suppose U is expressed as

$$U = 1 + \sum_i v_i(K)Bcw_i(K). \quad (\text{B.3})$$

If $1 + \sum_i v_i w_i \neq 0$, we can find its formal inverse³²,

$$U^{-1} = 1 - \frac{1}{1 + \sum_i v_i w_i} \sum_i v_i Bcw_i. \quad (\text{B.4})$$

The condition $1 + \sum_i v_i w_i \neq 0$ can be also expressed as

$$UB \neq 0. \quad (\text{B.5})$$

Therefore, Ψ can be written in the pure-gauge form (4.4) if

$$(1 + A\Psi)B \neq 0. \quad (\text{B.6})$$

³⁰As explained in section 4.1, we do not assume gauge parameters U and U^{-1} are regular.

³¹This procedure to render solutions into pure-gauge form is essentially the same as that presented by Ellwood in section 2.1 of [32].

³²When $1 + \sum_i v_i w_i = 0$, the string field U takes the following form:

$$U = - \sum_i v_i(K)cBw_i(K).$$

We do not consider its inverse, just as we do not consider the inverse of cB .

Next, we prove the inverse statement: if Ψ can be written in the pure-gauge form (4.4), then Ψ satisfies the equation (B.6). We can directly check the statement using the explicit expression (4.4).

To summarize, Ψ can be written in the pure-gauge form (4.4) if and only if

$$(1 + A\Psi)B \neq 0. \quad (\text{B.7})$$

Then, let us apply above discussion to the formal solution (5.5). Since (5.5) satisfies $(1 + A\Psi)B = 0$, it can not be written in the pure-gauge form (4.4). We note that since $(1 + A\Psi)B \neq 0$ for (5.4), it can be written in the pure-gauge form.

– **Discriminant function** $d_\Psi(x)$ Let us rephrase above discussion in terms of the discriminant function $d_\Psi(x)$ defined by

$$d_\Psi(K)B = (1 + A\Psi)B. \quad (\text{B.8})$$

The solution Ψ can be written in the form (4.4) if and only if $d_\Psi(K) \neq 0$. It is also true that Ψ is pure-gauge if and only if $d_\Psi(0) \neq 0, \infty$.

We notice that the discriminant function $d_\Psi(x)$ was used to classify formal pure-gauge solutions with respect to energy or boundary states. For Okawa's formal solutions (1.2), the discriminant function is

$$d_\Psi(x) = \frac{1}{1 - F(x)^2}, \quad (\text{B.9})$$

which was used in [36, 39] to classify the formal solutions by energy. For the formal solution (4.4), the discriminant function is given by

$$d_\Psi(x) = \frac{1}{1 - \sum_i I_i(x)J_j(x)}, \quad (\text{B.10})$$

which we used in section 4 to classify the formal solutions by boundary states. Indeed, both of these classifications are determined by the degree of poles or zeros of $d_\Psi(x)$ at $x = 0$.

B.2 On the classification of the formal pure-gauge solutions

In this subsection, we consider a group structure of ghost number zero string fields in the KBc subalgebra, which leads us to a classification of the formal pure-gauge solutions in terms of discriminant function. This classification may be regarded as an simple generalization of that given by Murata and Schnabl [36, 39]. It is also compatible with the discussion in section 4.2, which is classification of solutions under the regularity condition presented in section 2.

– **A class of gauge parameters** S_0 To begin with, we define a function $d_U(x)$, which is frequently used throughout the present subsection. Let U be a ghost number zero element in the KBc subalgebra. We define a function $d_U(x)$ for U as follows:

$$d_U(K)B = UB. \quad (\text{B.11})$$

The function d_U preserves multiplication of U 's:

$$d_{U_1 U_2} = d_{U_1} d_{U_2}. \quad (\text{B.12})$$

Using the function d_U , we define a class of gauge parameters S'_0 as

$$S'_0 = \left\{ U = 1 + \sum v_i B c w_i \mid d_U(x) \neq 0 \right\}, \quad (\text{B.13})$$

which is closed under star multiplication and inverses. It is important that (B.1) and (B.2) give one to one correspondence between the space of formal pure-gauge solutions (4.4) and gauge parameters S'_0 . In particular, any solution of the form (4.4) can be written as $\Psi = U^{-1}QU$, where $U \in S'_0$. In the following, we always keep this one to one correspondence in mind. Note that the discriminant function d_Ψ for $\Psi = U^{-1}QU$ ($U \in S'_0$) is given by d_U . For this reason, we also call $d_U(x)$ a discriminant function, as well as $d_\Psi(x)$.

For later use, we also define a class of gauge parameters S_0 as follows:

$$S_0 = \left\{ U \in S'_0 \mid d_U(x) \text{ is meromorphic around } x = 0. \right\}, \quad (\text{B.14})$$

which is also closed under multiplication and inverses.

– **Multiplicity** Since the discriminant function $d_U(x)$ for $U \in S_0$ is meromorphic around $x = 0$, we can expand $d_U(x)$ as follows:

$$d_U(x) = 1 + \sum_i v_i(x) w_i(x) = \sum_{m=n}^{\infty} a_m x^m. \quad (\text{B.15})$$

We define the multiplicity³³ n of a gauge parameter $U \in S_0$ by the integer n in (B.15). If U possesses multiplicity m and V possesses multiplicity n , then, from the property (B.12), the product UV possesses multiplicity $m+n$. We also note that any two gauge parameters U_1 and U_2 with the same multiplicity can be connected by a multiplicity zero gauge parameter. The class of multiplicity zero elements R make a normal subgroup of S_0 :

$$R = \{ U \in S_0 \mid d_U(0) \neq 0, d_U(0) \neq \infty \}. \quad (\text{B.16})$$

Since R is a normal subgroup of S_0 , we can consider the quotient group S_0/R , which is isomorphic to \mathbb{Z} .

³³ We expect that our multiplicity seems to be same as the winding number discussed by Hata and Kojita [44] up to some subtleties arising from regularization of solutions.

– **Classification of pure-gauge solutions by multiplicity** Now, let us consider a class of formal pure-gauge solutions whose discriminant functions $d_\Psi(x)$ are meromorphic around $x = 0$. This class of formal solutions correspond to elements of S_0 , and we define the multiplicity of such formal solutions by that of the corresponding gauge parameters $U \in S_0$. Any two solutions Ψ_1 and Ψ_2 of the same multiplicity can be connected by a multiplicity zero gauge parameter $U \in R$.

We note that if we impose our regularity condition on the formal pure-gauge solutions, the classification by multiplicity reduces to the discussion in section 4.2. Similarly, if we concentrate on the Okawa type solutions, the classification by multiplicity reduces to the discussion given by Murata and Schnabl [36, 39].

– **On the property of R** The classification by the multiplicity is natural from the algebraic point of view. Then, we would like to consider whether it has some physical meaning or whether it is related to gauge equivalence of solutions. Our question can be rephrased as whether the “gauge” transformations corresponding to elements of R possess some regular property or not, and, in the following, we give a naive discussion on their regularity.

Let U be an element of R . U can always be factorized into the following form:

$$U = U_1 * U_2 * \cdots * U_n, \quad (\text{B.17})$$

where each U_i takes the form

$$U_i = 1 + v_i(K) B c w_i(K). \quad (\text{B.18})$$

Functions v_i and w_i are meromorphic around the origin. Then, define \tilde{U}_i by

$$\tilde{U}_i = w_i(K) U_i w_i(K)^{-1} = 1 + v_i(K) w_i(K) B c. \quad (\text{B.19})$$

U can be written as

$$U = w_1^{-1} \tilde{U}_1 w_1 w_2^{-1} \tilde{U}_2 w_2 \cdots w_n^{-1} \tilde{U}_n w_n. \quad (\text{B.20})$$

The gauge transformation by U is equivalent to a reciprocal sequence of gauge transformation by \tilde{U}_i and similarity transformation $\Psi \rightarrow w_i^{-1} \Psi w_i$ or $\Psi \rightarrow w_i \Psi w_i^{-1}$. These similarity transformations are expected not to change energy of solutions. Then, **if** we omit all the w_i 's in (B.20), U reduces to \tilde{U} , where

$$\tilde{U} = \tilde{U}_1 * \tilde{U}_2 * \cdots * \tilde{U}_n. \quad (\text{B.21})$$

It is not difficult to see that the gauge parameter \tilde{U} is in the following form:

$$\tilde{U} = 1 + u(K) B c. \quad (\text{B.22})$$

Since $d_{U_i} = d_{\tilde{U}_i}$, it is also obvious that $d_{\tilde{U}} = 0$ and $\tilde{U} \in R$. Then, \tilde{U} and \tilde{U}^{-1} do not contain singular string fields like $1/K$. We naturally expect that gauge transformation by \tilde{U} is regular. We admit, however, that the above discussion is not sufficient to prove regularity of U .

– **Homomorphism of the KBc subalgebra** In the last of this appendix, we consider a series of formal homomorphisms h_g , which is a generalization of the first example given by Erler [57]. We define the action of h_g by³⁴

$$h_g(K) = g(K) \equiv \tilde{K}, \quad (\text{B.23})$$

$$h_g(B) = g(K)B/K \equiv \tilde{B}, \quad (\text{B.24})$$

$$h_g(c) = c(KB/g(K))c \equiv \tilde{c}, \quad (\text{B.25})$$

and

$$h_g(\Phi_1 * \Phi_2) = h_g(\Phi_1) * h_g(\Phi_2). \quad (\text{B.26})$$

Here the subscript g of h_g stands for the function $g(K)$. It is not difficult to confirm that \tilde{K} , \tilde{B} and \tilde{c} satisfy the same algebraic relations as K , B , and c :

$$\tilde{B}^2 = \tilde{c}^2 = 0, \quad \{\tilde{B}, \tilde{c}\} = 1, \quad Q\tilde{B} = \tilde{K}, \quad Q\tilde{c} = \tilde{c}\tilde{K}\tilde{c}, \quad [\tilde{K}, \tilde{B}] = 0, \quad (\text{B.27})$$

When the function $g(x)$ is meromorphic around $x = 0$, $h_g(S_0)$ is a subgroup of S_0 . We remark that multiplicity of an element of S_0 changes as $n \rightarrow mn$, where

$$m = \lim_{x \rightarrow 0} \frac{x d(\ln g(x))}{dx}. \quad (\text{B.28})$$

The action of h_g on the formal pure-gauge solutions takes quite simple form. Let Ψ_F be an Okawa type formal solution,

$$\Psi_F = F(K)c \frac{KB}{1 - F(K)^2} c F(K). \quad (\text{B.29})$$

Then the action of h_g is given by

$$h_g(\Psi_F) = \Psi_{\tilde{F}}, \quad (\text{B.30})$$

where

$$\tilde{F}(K) = F(\tilde{K}). \quad (\text{B.31})$$

Similarly, let $\Psi_{\{I_i, J_i\}}$ be a solution of the form (4.4). Then the action of h_g is

$$h_g(\Psi_{\{I_i, J_i\}}) = \Psi_{\{\tilde{I}_i, \tilde{J}_i\}}, \quad (\text{B.32})$$

where

$$\tilde{I}_i(K) = I_i(\tilde{K}), \quad \tilde{J}_i(K) = J_i(\tilde{K}). \quad (\text{B.33})$$

³⁴This generalization is also pointed out in [56] by Erler.

References

- [1] E. Witten, “Noncommutative Geometry and String Field Theory,” Nucl. Phys. B **268**, 253 (1986).
- [2] M. Schnabl, “Analytic solution for tachyon condensation in open string field theory,” Adv. Theor. Math. Phys. **10**, 433 (2006) [hep-th/0511286].
- [3] M. Schnabl, “Comments on marginal deformations in open string field theory,” Phys. Lett. B **654**, 194 (2007) [arXiv:hep-th/0701248].
- [4] M. Kiermaier, Y. Okawa, L. Rastelli and B. Zwiebach, “Analytic solutions for marginal deformations in open string field theory,” JHEP **0801**, 028 (2008) [hep-th/0701249 [HEP-TH]].
- [5] T. Erler, “Marginal Solutions for the Superstring,” JHEP **0707**, 050 (2007) [arXiv:0704.0930 [hep-th]].
- [6] Y. Okawa, “Analytic solutions for marginal deformations in open superstring field theory,” JHEP **0709**, 084 (2007) [arXiv:0704.0936 [hep-th]].
- [7] Y. Okawa, “Real analytic solutions for marginal deformations in open superstring field theory,” JHEP **0709**, 082 (2007) [arXiv:0704.3612 [hep-th]].
- [8] I. Ellwood, “Rolling to the tachyon vacuum in string field theory,” JHEP **0712**, 028 (2007) [arXiv:0705.0013 [hep-th]].
- [9] O. K. Kwon, “Marginally Deformed Rolling Tachyon around the Tachyon Vacuum in Open String Field Theory,” Nucl. Phys. B **804**, 1 (2008) [arXiv:0801.0573 [hep-th]].
- [10] S. Hellerman and M. Schnabl, “Light-like tachyon condensation in Open String Field Theory,” arXiv:0803.1184 [hep-th].
- [11] I. Kishimoto, “Comments on gauge invariant overlaps for marginal solutions in open string field theory,” Prog. Theor. Phys. **120**, 875 (2008) [arXiv:0808.0355 [hep-th]].
- [12] N. Barnaby, D. J. Mulryne, N. J. Nunes and P. Robinson, “Dynamics and Stability of Light-Like Tachyon Condensation,” JHEP **0903**, 018 (2009) [arXiv:0811.0608 [hep-th]].
- [13] F. Beaujean and N. Moeller, “Delays in Open String Field Theory,” arXiv:0912.1232 [hep-th].
- [14] M. Kiermaier, Y. Okawa and P. Soler, “Solutions from boundary condition changing operators in open string field theory,” JHEP **1103**, 122 (2011) [arXiv:1009.6185 [hep-th]].
- [15] T. Noumi and Y. Okawa, “Solutions from boundary condition changing operators in open superstring field theory,” JHEP **1112**, 034 (2011) [arXiv:1108.5317 [hep-th]].
- [16] I. Ellwood and M. Schnabl, “Proof of vanishing cohomology at the tachyon vacuum,” JHEP **0702**, 096 (2007) [hep-th/0606142].
- [17] T. Erler and M. Schnabl, “A Simple Analytic Solution for Tachyon Condensation,” JHEP **0910**, 066 (2009) [arXiv:0906.0979 [hep-th]].
- [18] Y. Okawa, “Comments on Schnabl’s analytic solution for tachyon condensation in Witten’s open string field theory,” JHEP **0604**, 055 (2006) [hep-th/0603159].

- [19] L. Rastelli and B. Zwiebach, “Tachyon potentials, star products and universality,” JHEP **0109**, 038 (2001) [hep-th/0006240].
- [20] M. Schnabl, “Wedge states in string field theory,” JHEP **0301**, 004 (2003) [hep-th/0201095].
- [21] L. Rastelli and B. Zwiebach, “Solving Open String Field Theory with Special Projectors,” JHEP **0801**, 020 (2008) [arXiv:hep-th/0606131].
- [22] Y. Okawa, L. Rastelli and B. Zwiebach, “Analytic Solutions for Tachyon Condensation with General Projectors,” arXiv:hep-th/0611110.
- [23] T. Erler, “Split String Formalism and the Closed String Vacuum,” JHEP **0705**, 083 (2007) [arXiv:hep-th/0611200].
- [24] T. Erler, “Split String Formalism and the Closed String Vacuum, II,” JHEP **0705**, 084 (2007) [arXiv:hep-th/0612050].
- [25] E. Fuchs, M. Kroyter and R. Potting, “Marginal deformations in string field theory,” JHEP **0709**, 101 (2007) [arXiv:0704.2222 [hep-th]].
- [26] E. Fuchs and M. Kroyter, “Marginal deformation for the photon in superstring field theory,” JHEP **0711**, 005 (2007) [arXiv:0706.0717 [hep-th]].
- [27] M. Kiermaier and Y. Okawa, “Exact marginality in open string field theory: a general framework,” JHEP **0911**, 041 (2009) [arXiv:0707.4472 [hep-th]].
- [28] T. Erler, “Tachyon Vacuum in Cubic Superstring Field Theory,” JHEP **0801**, 013 (2008) [arXiv:0707.4591 [hep-th]].
- [29] M. Kiermaier and Y. Okawa, “General marginal deformations in open superstring field theory,” JHEP **0911**, 042 (2009) [arXiv:0708.3394 [hep-th]].
- [30] I. Y. Aref’eva, R. V. Gorbachev and P. B. Medvedev, “Tachyon Solution in Cubic Neveu-Schwarz String Field Theory,” Theor. Math. Phys. **158**, 320 (2009) [arXiv:0804.2017 [hep-th]].
- [31] I. Y. Aref’eva, R. V. Gorbachev, D. A. Grigoryev, P. N. Khromov, M. V. Maltsev and P. B. Medvedev, “Pure Gauge Configurations and Tachyon Solutions to String Field Theories Equations of Motion,” JHEP **0905**, 050 (2009) [arXiv:0901.4533 [hep-th]].
- [32] I. Ellwood, “Singular gauge transformations in string field theory,” JHEP **0905**, 037 (2009) [arXiv:0903.0390 [hep-th]].
- [33] E. A. Arroyo, “Generating Erler-Schnabl-type Solution for Tachyon Vacuum in Cubic Superstring Field Theory,” J. Phys. A **43**, 445403 (2010) [arXiv:1004.3030 [hep-th]].
- [34] T. Erler, “Exotic Universal Solutions in Cubic Superstring Field Theory,” JHEP **1104**, 107 (2011) [arXiv:1009.1865 [hep-th]].
- [35] L. Bonora, C. Maccaferri and D. D. Tolla, “Relevant Deformations in Open String Field Theory: a Simple Solution for Lumps,” arXiv:1009.4158 [hep-th].
- [36] M. Murata and M. Schnabl, “On Multibrane Solutions in Open String Field Theory,” Prog. Theor. Phys. Suppl. **188**, 50 (2011) [arXiv:1103.1382 [hep-th]].

- [37] L. Bonora, S. Giaccari and D. D. Tolla, “The energy of the analytic lump solution in SFT,” JHEP **1108**, 158 (2011) [arXiv:1105.5926 [hep-th]].
- [38] T. Erler and C. Maccaferri, “Comments on Lumps from RG flows,” arXiv:1105.6057 [hep-th].
- [39] M. Murata and M. Schnabl, “Multibrane Solutions in Open String Field Theory,” arXiv:1112.0591 [hep-th].
- [40] A. Sen, “Universality of the tachyon potential,” JHEP **9912**, 027 (1999) [hep-th/9911116].
- [41] A. Hashimoto and N. Itzhaki, “Observables of string field theory,” JHEP **0201**, 028 (2002) [hep-th/0111092].
- [42] D. Gaiotto, L. Rastelli, A. Sen and B. Zwiebach, “Ghost structure and closed strings in vacuum string field theory,” Adv. Theor. Math. Phys. **6**, 403 (2003) [hep-th/0111129].
- [43] I. Ellwood, “The Closed string tadpole in open string field theory,” JHEP **0808**, 063 (2008) [arXiv:0804.1131 [hep-th]].
- [44] H. Hata and T. Kojita, “Winding Number in String Field Theory,” JHEP **1201**, 088 (2012) [arXiv:1111.2389 [hep-th]].
- [45] T. Masuda, “On the classical solution for the double-brane background in open string field theory,” *to appear*.
- [46] D. Takahashi, “The boundary state for a class of analytic solutions in open string field theory,” JHEP **1111**, 054 (2011) [arXiv:1110.1443 [hep-th]].
- [47] M. Kiermaier, Y. Okawa and B. Zwiebach, “The boundary state from open string fields,” arXiv:0810.1737 [hep-th].
- [48] M. Kudrna, C. Maccaferri and M. Schnabl, “Boundary State from Ellwood Invariants,” arXiv:1207.4785 [hep-th].
- [49] T. Okuda and T. Takayanagi, “Ghost D-branes,” JHEP **0603**, 062 (2006) [hep-th/0601024].
- [50] M. Kiermaier, A. Sen and B. Zwiebach, “Linear b-Gauges for Open String Fields,” JHEP **0803**, 050 (2008) [arXiv:0712.0627 [hep-th]].
- [51] M. Kiermaier and B. Zwiebach, “One-Loop Riemann Surfaces in Schnabl Gauge,” JHEP **0807**, 063 (2008) [arXiv:0805.3701 [hep-th]].
- [52] T. Erler and C. Maccaferri, “Connecting Solutions in Open String Field Theory with Singular Gauge Transformations,” JHEP **1204**, 107 (2012) [arXiv:1201.5119 [hep-th]].
- [53] T. Erler and C. Maccaferri, “The Phantom Term in Open String Field Theory,” JHEP **1206** (2012) 084 [arXiv:1201.5122 [hep-th]].
- [54] S. Zeze, “Regularization of identity based solution in string field theory,” JHEP **1010**, 070 (2010) [arXiv:1008.1104 [hep-th]].
- [55] E. A. Arroyo, “Comments on regularization of identity based solutions in string field theory,” JHEP **1011**, 135 (2010) [arXiv:1009.0198 [hep-th]].

- [56] T. Erler, “The Identity String Field and the Sliver Frame Level Expansion,” arXiv:1208.6287 [hep-th].
- [57] T. Erler, “A simple analytic solution for tachyon condensation,” Theor. Math. Phys. **163**, 705 (2010) [Teor. Mat. Fiz. **163**, 366 (2010)].